# Dehn Functions and the Isoperimetric Spectrum

by

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**MA4K9 Dissertation**

Submitted to The University of Warwick

**Mathematics Institute**

April, 2024

# Contents

# 1 Introduction

The main goal of this work is to show that there exists a unique gap in the *isoperimetric spectrum*, $\mathbb{P}$, defined as follows:

$$\mathbb{P} = \{\alpha \in [1, \infty) : f(n) = n^\alpha \text{ is a Dehn function}\}.$$

This gap, which we refer to throughout this work as the *Gromov gap*, is the interval $(1, 2) \subset \mathbb{P}$. This name arises from Gromov, who first proved its existence [1] in 1987. In the early 1990s, Gromov's result prompted research on the isoperimetric spectrum, which was pioneered by Brady and Bridson among others. The question of whether or not there exist groups whose Dehn functions are of rational (and later transcendental and irrational) degree attracted interest to many geometric group theorists, and this topic continues to be an active area of research in Geometric Group Theory today.

In order to delve into the isoperimetric spectrum, we first introduce some preliminaries about Dehn functions in Section 2, and we relate both the algebraic and topological sides of the story. In particular, our exposition is guided by Bridson's paper, *The Geometry of the Word Problem* [2] for the algebraic ideas and a combination of Brady-Riley-Short's text, *The Geometry of the Word Problem for Finitely Generated Groups* [3] and Bridson-Haefliger's *Metric Spaces of Non-Positive Curvature* [4] for the topological concepts.

In Section 3, we prove that if a group has a subquadratic Dehn function, then it in fact has a linear Dehn function. Although orginally shown by Gromov (1987), here we combine the approaches of Papasoglu [5] and Bridson-Haefliger [5, p. 417–8] to prove the existence of this gap in the isoperimetric spectrum.

We then follow the work of Brady and Bridson in their paper titled, *There is only one gap in the isoperimetric spectrum* [6] in Section 4. We touch upon snowflake groups as a graph of groups and use this construction to determine a lower and upper bound for the Dehn function of this family of groups. After these proofs, we eventually conclude that the Gromov gap is unique.

To conclude, we mention some related results about the isoperimetric spectrum which led to this discovery, as well as some more recent work, in our concluding remarks. As far as possible, this dissertation is self-contained, however for brevity we take some results as a black box. In particular, results which would otherwise introduce a tangent are listed in the Appendix.

Finally, thank you to my supervisor Dr. Robert Kropholler for all his guidance on this topic and his help on some (very!) challenging proofs.

# 2 Background

## 2.1 Algebraic notions

In order to define Dehn functions and the isoperimetric spectrum, we first introduce some key ideas and notation for our discussion. We recall the definition of the free group from *MA4H4 Geometric Group Theory*.

**Definition 2.1.1** (Free group, equivalent words)**.** The *free group* on the set $S$, denoted $F(S)$, consists of the set of equivalence classes of words constructed from the alphabet $S \cup S^{-1}$. The words $w_1, w_2 \in F(S)$ are *equivalent*, written $w_1 \sim w_2$, if there is a sequence of elementary contractions or expansions between the words $w_1$ and $w_2$.

We assume that the free group is in fact a group, and therefore that compositions within the free group are well-defined and so every word corresponds to a unique reduced word.

Next we introduce two different types of closure for a subset of a group $G$.

**Definition 2.1.2** (Normal closure)**.** Let $G$ be a group. The *normal closure* of a subset $A \subseteq G$ is the unique smallest normal subgroup of G containing A, denoted by $\langle\langle A \rangle\rangle$. Explicitly, $\langle\langle A \rangle\rangle := \langle g^{-1}ag : g \in G, a \in A \rangle$.

**Definition 2.1.3** (Symmetric closure)**.** The *symmetric closure* of the set of relations $R$, denoted by $R_*$, is the set of elements of $R$, their cyclic permutations and the cyclic permutations of their inverses.

This leads us to the following standard definition for the presentation of a group.

**Definition 2.1.4** (Presentation of a group)**.** Let $G$ be a group. A *presentation* $\mathcal{P}$ of $G$ is defined by $\langle S \mid R \rangle = F(S)/\langle\langle R \rangle\rangle$.

In this section, we take $G = \langle S \mid R \rangle$ as a finitely-presented group with generating set $S$ and relations $R$. For the free group $F(S)$, we write $\mu : F(S) \to G$ as the natural surjection of the free group generated by $S$ onto $G$. That is, for any word $w$ in the free group $F(S)$, $\mu(w) \in G$ is the unique reduced word.

**Notation 2.1.5.** *Let $w_1$ and $w_2$ be not necessarily reduced words in $F(S)$. We follow Bridson's article [2, p. 5] and introduce the following notation.*

- *We write $w_1 \underset{F(S)}{=} w_2$ when the words $w_1, w_2$ are equal in the free group, i.e. they are elements of the same equivalence class (equal up to elementary expansions and contractions). Sometimes, we write this as $w_1 = w_2$ in $F(S)$, and these two notations are interchangeable. For $w_1 \underset{G}{=} w_2$, we simply write $w_1 = w_2$.*
- *We write $w_1 \equiv w_2$ when the words are identical.*

**Example 2.1.6.** Consider the group presentation $G = \langle x, y \mid x^2 \rangle$. Let $w \equiv xyxx \in F(S)$. Then $\mu(w) = xyx^2 = xy \in G$.

**Definition 2.1.7** (Null-homotopic)**.** A word $w$ (not necessarily reduced) in $F(S)$ is *null-homotopic* if $\mu(w) = 1$.

**Lemma 2.1.8.** *A word $w$ in $F(S)$ is null-homotopic if and only if it can be written in the form*

$$w \underset{F(S)}{=} \prod_{i=1}^{n} u_i r_i^{\pm 1} u_i^{-1}$$

*where $u_i \in F(S)$ and $r_i \in R$, for $i \in \{1, \dots, n\}$.*

***Proof.*** The reverse implication is trivial by free cancellation, so we concentrate on the forward implication. By Definition 2.1.2, the elements of $G$ which can be expressed in the form $\prod_{i=1}^{n} u_i r_i^{\pm 1} u_i^{-1}$ described above are precisely the elements of the normal closure $\langle\langle R \rangle\rangle$. Hence, it suffices to show that if $w$ is null-homotopic, then $w \in \langle\langle R \rangle\rangle$. Suppose not, then there is $w \in \langle\langle R \rangle\rangle$ such that $w$ is not null-homotopic. But each generator of the normal closure of $R$ is of the form $u_i r u_i^{-1} = u_i u_i^{-1} = 1_G$, which contradicts the assumption that $w$ is not null-homotopic. □

**Definition 2.1.9** (Area of a word)**.** [7, p. 35] Under the above conditions, the *area* of a word $w$ is defined to be

$$A(w) = \min\{n \in \mathbb{N} : w \underset{F(S)}{=} \prod_{i=1}^{n} u_i r_i^{\pm 1} u_i^{-1}\}.$$

In other words, it is the smallest number $n$ for which a word $w$ equivalent to the identity element can be expressed as the product of $n$ conjugates of relators (and their inverses). From this we observe the following useful fact.

**Lemma 2.1.10.** *If $w_1$ and $w_2$ are null-homotopic words in $F(S)$, then $A(w_1 w_2) \leq A(w_1) + A(w_2)$.*

***Proof.*** The proof of this arises by concatenating the two minimal products of conjugates of relations for the words $w_1$ and $w_2$ and observing that this is a product of conjugates of relations (not necessarily minimal) for the product $w_1 w_2$. □

We next introduce the Dehn function, which measures the minimal number of defining relations needed to reduce a word to the identity element. The primary reason that it is of interest is because it provides us with a quantitative measure of complexity of the *Word Problem.* That is to say, the Dehn function gives us an answer to the following question: *Given an word $w$ formed from the set of generators and their inverses of a finitely-presented group $\langle S \mid R \rangle$, how difficult is it to determine whether $w$ represents the identity element in this group?*

**Definition 2.1.11** (Dehn function, algebraic)**.** The *Dehn function* for a presentation

$\langle S \mid R \rangle$ is a map $\delta : \mathbb{N} \to \mathbb{N}$ defined by

$$\delta(n) = \left\{ \max_{w \in F(S)} A(w) : \ell(w) \leq n,\ \mu(w) = 1 \right\}.$$

Sometimes we denote the Dehn function with respect to a particular presentation $\mathcal{P} = \langle S \mid R \rangle$ by $\delta_{\mathcal{P}}$.

We briefly remark that the map $\delta$ is well-defined since $\langle S \mid R \rangle$ is finitely-presented. Therefore for any fixed $n \in \mathbb{N}$, the value of $\delta(n)$ is finite since there are only finitely many words of length at most $n$ [6, p. 2].

**Definition 2.1.12** (Dehn dominated, Dehn equivalent)**.** Given two functions $f, g : [0, \infty) \to [0, \infty)$, we say that $f$ is *Dehn dominated* by $g$, denoted $f \preceq g$, if there exists a constant $K > 0$ such that

$$f(x) \leq Kg(Kx + K) + Kx + K \text{ for all } x \in [0, \infty).$$

If $f \preceq g$, and $g \preceq f$, then we say that $f$ and $g$ are *Dehn equivalent* and denote this by $f \simeq g$.

**Lemma 2.1.13.** *The relation $\simeq$ as above is an equivalence relation.*

***Proof.*** This is straightforward to prove in the usual way, and therefore we omit this proof. □

A natural question that arises at this stage is: *Which relations between groups $G, H$ guarantee that the groups have equivalent Dehn functions?* One answer to this is that a change of group presentation for the same group leads to an equivalent Dehn function. This provides us with a useful trade-off: we can think about the Dehn function of a group up to equivalence with the upshot that it is independent of the group presentation.

**Proposition 2.1.14.** *Let $G, H$ be isomorphic finitely-presented groups with presentations $\mathcal{P}, \mathcal{Q}$ respectively. Then the Dehn functions of those presentations are equivalent, i.e. $\delta_{\mathcal{P}} \simeq \delta_{\mathcal{Q}}$.*

***Proof.*** We outline the reasoning provided in Bridson's paper [2, p. 7], using the notes [8, p. 15–16]. To start, we consider two special cases. We then show that every presentation can be derived from these two cases.

*Case* 1. $R' \subseteq \langle\langle R \rangle\rangle$ is a finite set and $\mathcal{P} = \langle S \mid R \rangle$, $\mathcal{Q} = \langle S \mid R \cup R' \rangle$.

In this case, we show that adding redundant relators $R'$ to a group presentation results in an equivalent Dehn function. This means that each relator $r \in R'$ can be expressed in $F(S)$ as a product of $m_r$ conjugates of the old relators $R$ and their inverses. Let $m$ be the maximum of the $m_r$ for all $r \in R'$, and suppose that a word $w \in F(S)$ can be written as a product of $N$ conjugates of relators from $R \cup R'$ and their inverses.

By this, we can rewrite $w$ as a product of at most $mN$ conjugates of the relators $R^{\pm 1}$. Notice also that by definition, the area of $w$ with respect to $\mathcal{Q}$ is at most its area with respect to $\mathcal{P}$. Therefore, we deduce that $\delta_{\mathcal{Q}}(n) \leq \delta_{\mathcal{P}}(n) \leq m\delta_{\mathcal{Q}}(n)$ for all $n \in \mathbb{N}$. Hence, $\delta_{\mathcal{P}} \simeq \delta_{\mathcal{Q}}$.

*Case* 2. $\mathcal{P} = \langle S \mid R \rangle$ and $\mathcal{Q} = \langle S \cup T \mid R \cup R' \rangle$, where $R' = \{tw_t^{-1} : t \in T, w_t \in F(S)\}$.

We first show that $\delta_{\mathcal{P}} \preceq \delta_{\mathcal{Q}}$. Consider the retraction $\rho : F(S \cup T) \to F(S)$ which sends $s \mapsto s$ and $t \mapsto w_t$. Observe that $\rho(r) = r$ for all $r \in R$, and $\rho(r') = 1_{F(S)}$ for all $r' \in R'$.

Let $w \in F(S)$ and $d$ be the area of $w$ with respect to the presentation $\mathcal{Q}$, denoted $A_{\mathcal{Q}}(w)$. Then, $w = \prod_{i=1}^{d} u_i r_i^{\pm 1} u_i^{-1}$ where $g_i \in F(S \cup T)$, and $r_i \in R \cup R'$. Applying the retraction $\rho$, $\rho(w) = \prod_{i=1}^{d} \rho(u_i)\rho(r_i)^{\pm 1}\rho(u_i^{-1})$ and $\rho(w) = w$ since $\rho$ is a retraction.

Therefore, we have

$$w = \prod_{i=1}^{d} \rho(u_i)\rho(r_i)^{\pm 1}\rho(u_i^{-1}).$$

By our observation, $\rho(r_i)$ equals $r_i$ or $1_{F(S)}$ for each $1 \leq i \leq d$, $w$ can be written as product of at most $d$ conjugates of relators and their inverses in $R$. Hence $\delta_{\mathcal{P}} \preceq \delta_{\mathcal{Q}}$.

Next, we show $\delta_{\mathcal{Q}} \preceq \delta_{\mathcal{P}}$. Let $M = \max_{t \in T}\{\ell_S(w_t)\}$, where $\ell_S$ is the length with respect to the generating set $S$. Consider a null-homotopic word $w \in F(S \cup T)$. By applying relations from $R'$, we can write this as a word $w' \in F(S)$. Explicitly,

$$\begin{aligned} w &= s_{i_1}^{\pm 1} s_{i_2}^{\pm 1} \dots t_{j_1} \dots s \dots t \dots \in F(S \cup T) \\ w' &= s_{i_1}^{\pm 1} s_{i_2}^{\pm 1} \dots w_{t_{j_1}} \dots s \dots w_t \dots \in F(S). \end{aligned}$$

From this, $\ell_S(w') \leq M\ell_{S \cup T}(w)$. Since we apply at most $\ell_{S \cup T}(w)$ relators to transform $w$ into $w'$, we have that

$$\begin{aligned} A_{\mathcal{Q}}(w) &\leq A_{\mathcal{P}}(w') + \ell_{S \cup T}(w) \\ &\leq \delta_{\mathcal{P}}(M\ell_{S \cup T}(w)) + \ell_{S \cup T}(w). \end{aligned}$$

It follows that $\delta_{\mathcal{Q}}(n) \leq \delta_{\mathcal{P}}(Mn) + n$ and so $\delta_{\mathcal{Q}} \preceq \delta_{\mathcal{P}}$.

*Proof of proposition:* Suppose the presentations $\mathcal{P} = \langle S \mid R \rangle$ and $\mathcal{Q} = \langle S' \mid R' \rangle$ give rise to isomorphic groups. Then for all $s \in S$, there is $u_s \in F(S')$ such that $s$ maps to $u_s$ under the isomorphism between $\mathcal{P}$ and $\mathcal{Q}$. Similarly, for all $s' \in S'$, there is a corresponding element $v_{s'} \in F(S)$.

Let $T = \{su_s^{-1} : s \in S\}$ and $T' = \{s'v_{s'}^{-1} : s' \in S'\}$. We then define a presentation $\mathcal{I} = \langle S \cup S' \mid R \cup R' \cup T \cup T' \rangle$ which acts as an intermediate presentation between $\mathcal{P}$

and $\mathcal{Q}$ we can transform to using the above two cases. Therefore, $\delta_{\mathcal{P}} \simeq \delta_{\mathcal{I}} \simeq \delta_{\mathcal{Q}}$ and by transitivity of the equivalence relation $\simeq$, we are done. □

A more general result (which we will not prove) is that the Dehn function is actually a quasi-isometry invariant, up to equivalence.

**Theorem 2.1.15.** *If $G$ is a finitely presented group with presentation $\mathcal{P}$ and $G'$ is a finitely generated group quasi-isometric to $G$, then $G'$ is also finitely presented with finite presentation $\mathcal{Q}$ and the Dehn functions of $G$ with respect to $\mathcal{P}$ and $G'$ with respect to $\mathcal{Q}$ are equivalent, i.e. $\delta_{\mathcal{P}} \simeq \delta_{\mathcal{Q}}$.*

## 2.2 Topological notions

### 2.2.1 Van Kampen diagrams

We now turn our attention to the geometrical notion of isoperimetry. Informally, the isoperimetric problem concerns finding a closed curve with the smallest perimeter which encloses the largest area. To see how Dehn functions relate to this intuitive concept of isoperimetry, it is useful to visualise the area of a word, which we introduced in Definition 2.1.9. To do this, we use a construction known as a van Kampen diagram. We proceed to define this construction formally by introducing some new notation and definitions from [3, p. 89–90]. We then return to defining isoperimetry more formally in Section 2.3.

**Definition 2.2.1** (Combinatorial, combinatorial complex)**.** A continuous map between CW complexes is said to be *combinatorial* if it sends open cells homeomorphically onto open cells. A *combinatorial complex* is a CW complex all of whose attaching maps are combinatorial (with respect to some CW structure on the boundary of the cell being attached).

**Definition 2.2.2** (Presentation 2-complex)**.** Let $G$ be a finitely presented group with presentation $\mathcal{P} = \langle S \mid R \rangle = \langle x_1, \dots, x_m \mid r_1, \dots, r_n \rangle$. We consider the following construction.

1. Take $m$ oriented edges, labelled by the generators $x_1, \dots, x_m$, and identify all the vertices to form a rose.

2. Let $C_1, \dots, C_n$ be 2-cells, where each 2-cell has $C_i$ has $n_i$ edges, where $n_i$ is the length of the relator $r_i$. Attach these 2-cells by identifying the boundary circuit of $C_i$ with the edge-path in the rose along which one reads $r_i$.

The resulting complex is called the *presentation* 2*-complex* and is denoted $K_{\langle S|R \rangle}$.

**Definition 2.2.3** (Cayley 2-complex)**.** The *Cayley* 2*-complex*, denoted $\mathrm{Cay}^2(\mathcal{P})$, is the universal cover of the presentation 2-complex for $\mathcal{P}$.

**Definition 2.2.4** (van Kampen diagram)**.** A *van Kampen diagram* over a presentation $\langle S \mid R \rangle$ is combinatorial map $\Delta \to \mathrm{Cay}^2(\mathcal{P})$, where $\Delta$ is a connected, simply connected, planar 2-complex[1], which satisfies the following property: For each 2-cell of $\Delta$, the label of the boundary cycle of the region, is a reduced word in $F(S)$ that belongs to $R_*$.

From this, the diagram has a boundary cycle, denoted by $\partial\Delta$, which is an edge loop in the 1-skeleton starting and ending at the base vertex of $\Delta$ and going around $\Delta$ in the clockwise direction. The label of the boundary cycle is a null-homotopic word $w \in F(S)$ called the *boundary label*. We say that $\Delta$ is a van Kampen diagram for $w$.

Pictorially, we have that each 1-cell of $\Delta$ is labelled by an arrow and a letter $s \in S$, and some vertex in the topological boundary of $\Delta \subseteq R^2$ is a base-vertex[2].

**Example 2.2.5.** To see this, consider the van Kampen diagram below.

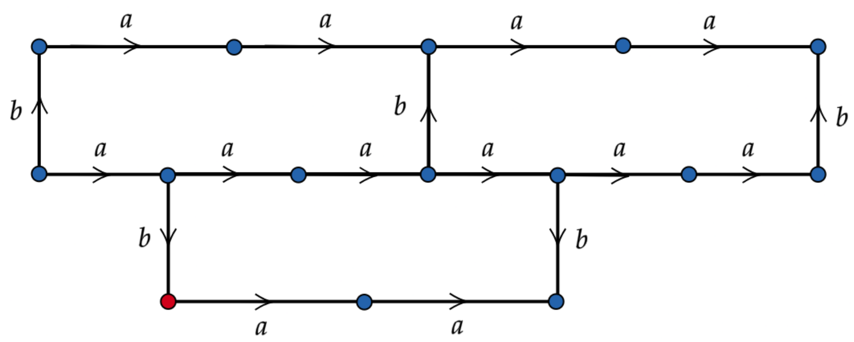


Figure 1: van Kampen diagram $\Delta$ for the presentation $\langle a, b \mid ba^2b^{-1}a^{-3} = 1\rangle$. The base-vertex is indicated by the red vertex as shown. Figure replicated from [9, p. 7].

Reading off the diagram, $\Delta$ is a van Kampen diagram for the word $w = a^2b^{-1}a^2ba^{-4}b^{-1}ab$ represented by the boundary cycle.

We check that the diagram $\Delta$ is a van Kampen diagram. It is simply connected as there is a path between any two vertices, and every loop is homotopy equivalent to a point since each loop contains a 2-cell. Every 1-cell is labelled by an arrow and a generator, and the base-vertex is marked by the vertex in red.

It is quick to check that each of the three 2-cells has boundary word which is a cyclic permutation of the relator $r = ba^2b^{-1}a^{-3}$ or its inverse $r^{-1} = a^3ba^{-2}b^{-1}$. The set of cyclic permutations of $r$ and cyclic permutations of $r^{-1}$ is exactly the symmetric closure $R_*$.

We next establish some key properties and characteristics of van Kampen diagrams.

**Definition 2.2.6** (Minimal van Kampen diagram)**.** For a van Kampen diagram $\Delta$ of a word $w$, we call the diagram $\Delta$ *minimal* if it has the minimum total number of cells of any diagram for $w$.

[1]Equivalently, some sources define $\Delta$ equal to $S^2 \setminus e_\infty$, for $S^2$ homeomorphic to $\mathbb{S}^2$ and $e_\infty$ an open 2-cell of $S^2$.

[2]More precisely, a base-vertex is a basepoint of the topological space $\Delta$, where we specify this point to be a vertex of the graph given by the 1-skeleton of $\Delta$.

**Definition 2.2.7** (Area of a van Kampen diagram)**.** The *area of a van Kampen diagram* $\Delta$, denoted $A(\Delta)$, is defined by the number of 2-cells in $\Delta$.

The following result on the existence of van Kampen diagrams is a key result often referred to as van Kampen's Lemma. It allows us to relate the algebraic idea of a null-homotopic word in a free group to a van Kampen diagram corresponding to that word, making them a useful tool for studying isoperimetric functions.

**Lemma 2.2.8** (van Kampen's Lemma)**.** *Let $G = \langle S \mid R \rangle$ and $w \in F(S)$. The following the equivalent:*

*(i) The word $w$ is null-homotopic.*

*(ii) There exists a van Kampen diagram $\Delta$ over the presentation $\langle S \mid R \rangle$ where $w$ is the boundary label.*

***Proof.*** [Sketch] For the full details, we refer the reader to Bridson's article [2, Section 4].

To show that (i) implies (ii), recall that by Lemma 2.1.8, if $w$ is null-homotopic in $F(S)$, then it can be expressed as a product $w \underset{F(S)}{=} \prod_{i=1}^{n} u_i r_i^{\pm 1} u_i^{-1}$ where $u_i \in F(S)$ and $r_i \in R$, for $i \in \{1, \ldots, n\}$. We construct the *lollipop diagram* whose boundary word is the unreduced product given by $\prod_{i=1}^{n} u_i r_i^{\pm 1} u_i^{-1}$.

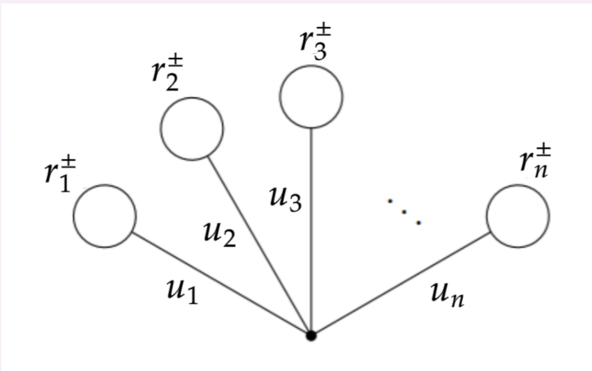


Figure 2: Lollipop diagram for a word $w$ written as a product of $n$ conjugates of relators (and their inverses).

Recall that $w$ is not necessarily a reduced word, and this was highlighted by the distinction in our notation $\underset{F(S)}{=}$ and $\equiv$. Therefore, this product may not be exactly equal to the word $w$, but can be transformed into the word $w$ by a finite sequence of moves consisting of elementary contractions and expansions. In this way, it follows that the lollipop diagram can be modified by a finite sequence of moves to produce a van Kampen diagram with boundary label $w$.

To show that (ii) implies (i), we may proceed by induction on the area of the van

Kampen diagram $\Delta$. For $n = 1$, let $e$ be the 2-cell that the boundary curve $w$ arrives at (reading anti-clockwise around $\partial\Delta$). Let $g$ be the path from the basepoint vertex $p$ to $e$, and let $r$ be the relator read anti-clockwise around $e$. Then, $w \equiv grg^{-1}$, where $g$ is a product of elements in $S \cup S^{-1}$. Then, it follows that $w \equiv grg^{-1} = gg^{-1} = 1_G$ in $G$. For the inductive step, let $e$ be the first 2-cell that the boundary curve $w$ arrives at, and let $\Delta' = \Delta \setminus e$. Since $A(\Delta') = A(\Delta) - 1$, we can see by induction that $w \underset{F(S)}{=} \prod_{i=1}^{n} u_i r_i^{\pm 1} u_i^{-1}$ where $n = A(\Delta)$. □

Due to this lemma, the Word Problem effectively reduces to determining whether a word admits a van Kampen diagram. This has a useful corollary as follows.

**Corollary 2.2.9.** *Let $\Delta$ be a minimal van Kampen diagram with boundary word $w$. Then the area of the van Kampen diagram is equal to the area of the word $w$, i.e. $A(\Delta) = A(w)$.*

***Proof.*** As a consequence of the proof of Lemma 2.2.8, for a diagram $\Delta$ with boundary word $w$, $A(\Delta)$ is equal to the number of conjugates of relators whose product gives the word $w$ [2, p. 20-21]. By modifying the van Kampen diagram using the lollipop diagram as above, the area of a minimal van Kampen diagram for $w$ corresponds a minimal number of conjugates of relators whose product gives $w$. Therefore, $A(\Delta) = A(w)$. For further details, see [4, p. 155]. □

### 2.2.2 Area fillings

In this section, we continue to define topological notions and progress to dealing with metric graphs and $\varepsilon$-fillings. In doing this, we lay out the groundwork for proving Theorem **B** in Section 3.2. Here we use definitions provided in Bridson and Haefliger [4, p. 414].

To start we introduce the notion of a metric graph, which is a type of length space.

**Definition 2.2.10** (Length metric, length space)**.** Let $(X, d)$ be a metric space. A metric $d$ is said to be a *length metric* if the distance between every pair of points $x, y \in X$ is equal to the infimum of the length of rectifiable curves joining them. If there are no such curves then $d(x, y) = \infty$. If $d$ is a length metric then the metric space $(X, d)$ is called a *length space*.

The formal construction of a metric graph is fairly involved, so we outline a more intuitive definition which will be more useful in our approach. We may think of metric graphs as metric spaces that we may obtain by taking a connected graph (i.e. a connected 1-dimensional CW-complex), and applying a notion of distance to the edges of the graph as bounded intervals of the real line. We define the metric between two points by the length metric. In this context, this metric is defined by the infimum of the lengths of paths joining the points, where length is measured using the chosen metrics along the edges.

In the context of metric graphs, we define useful maps including triangulations and more specifically $\varepsilon$-fillings.

**Definition 2.2.11** (Triangulation)**.** A *triangulation* of the unit disk $\mathbb{D}^2$ is a homeomorphism $P : \mathbb{D}^2 \to X$, where X is a combinatorial 2-complex in which every 2-cell is has three edges.

We endow $\mathbb{D}^2$ with the induced cell structure and refer to the pre-images under $P$ of 0-cells, 1-cells and 2-cells as, respectively, the *vertices*, *edges* and *faces* of $P$.

**Definition 2.2.12** ($\varepsilon$-filling, $\varepsilon$-area)**.** Let $X$ be a metric space and $\gamma : \mathbb{S}^1 \to X$ be a rectifiable[3] loop in X. An *$\varepsilon$-filling* $(P, \Phi)$ of $\gamma$ consists of a triangulation $P$ of $\mathbb{D}^2$ and a (not necessarily continuous) map $\Phi : \mathbb{D}^2 \to X$ such that $\Phi|_{\mathbb{S}^1} = \gamma$ and the image under $\Phi$ of each face of $P$ is a set of diameter at most $\varepsilon$. We write $|\Phi|$ to denote the number of faces of $P$ and refer to this as the *area of the filling*. The *$\varepsilon$-area* of $\gamma$ is defined to be:

$$\text{Area}_\varepsilon(\gamma) = \min\left\{|\Phi| : \Phi \text{ is an } \varepsilon\text{-filling of } \gamma\right\}.$$

If there is no $\varepsilon$-filling for a given value of $\varepsilon$, then we define $\text{Area}_\varepsilon(\gamma) \coloneqq \infty$.

**Definition 2.2.13** (Edge-loop)**.** Let $X$ be a metric graph. An *edge-loop* in $X$ is a loop $\gamma : \mathbb{S}^1 \to X$ which is the concatenation of a finite number of paths, each of which is either:

- ⋆ A constant speed parametrisation of an edge: denoted by $\gamma_i : \mathbb{S}^1 \to X$.
- ⋆ A constant map at a vertex, denoted by $c_j : v_j \to X$ for $v_j \in \mathcal{V}$, where $\mathcal{V}$ represents the vertex set of the graph $X$.

We write $l_0(\gamma)$ to denote the number of maximal non-trivial arcs where $\gamma : \mathbb{S}^1 \to X$ is constant.

*Remark* 2.2.14. Observe that $l_0(\gamma) \leq l(\gamma) + 1$. We have equality in the case where $\gamma$ is the constant loop at a point in $X$.

This definition gives rise to the definition of an $\varepsilon$-filling of an edge-loop, called a standard $\varepsilon$-filling. This is formally defined by applying the definition above to an edge-loop in way we might expect.

**Definition 2.2.15** (Standard $\varepsilon$-filling)**.** A *standard $\varepsilon$-filling* of an edge-loop $\gamma$ is an $\varepsilon$-filling $(P, \Phi)$ given by a triangulation $P$ of the disk such that all of the vertices on the boundary circle $\partial\mathbb{D}^2$ are points of concatenation of the given edge-loop $\gamma$, and each edge of the triangulation is either mapped to a concatenation of edges in $X$, or is sent to a vertex of $X$ by a constant map.

[3] Loosely speaking, this means that the loop has finite length.

## 2.3 Isoperimetric notions

We now relate our work so far to isoperimetry by recalling the following key definition.

**Definition 2.3.1** (Isoperimetric spectrum)**.** The *isoperimetric spectrum* is defined by the set

$$\mathbb{P} = \{\alpha \in [1, \infty) : f(n) = n^{\alpha} \text{ is a Dehn function}\}.$$

We will later prove the following theorems, and conclude that linear Dehn function characterises hyperbolic groups by proving the two theorems below. In both cases, we take the group $G$ to be finitely-presented, and fix a presentation $\langle S \mid R \rangle$ for $G$.

**Theorem A.** *If $\delta$ is subquadratic for the presentation $\langle S \mid R \rangle$, then $G$ is hyperbolic.*

**Theorem B.** *If $G = \langle S \mid R \rangle$ is hyperbolic, then $\delta$ is linear for the presentation of $G$.*

Together, these results implies the unexpected result that the isoperimetric spectrum $\mathbb{P}$ contains a gap which is exactly the interval $(1, 2)$. This is often referred to as the *Gromov Gap*.

**Theorem C.** *More precisely, if $\delta$ is subquadratic for the presentation $\langle S \mid R \rangle$, then $\delta$ is in fact linear, and so $\mathbb{P} \cap (1, 2) = \varnothing$.*

As shown in Brady and Bridson's paper [6], it turns out that this is the only gap in the isoperimetric spectrum. We discuss the techniques from [6] used to prove this result in Section 4.

**Definition 2.3.2** (Isoperimetric inequality for groups)**.** Let $G$ be a group.

- ⋆ We say that $G$ satisfies a *linear isoperimetric inequality* if there exists $k \in \mathbb{R}$ such that $A(w) \leq k \cdot \ell(w)$ for all the words $w \in F(S)$ with $\mu(w) = 1$.
- ⋆ Similarly, we say that $G$ satisfies a *subquadratic isoperimetric inequality* if there exists $k \in \mathbb{R}$ such that $A(w) < k \cdot \ell(w)^2$ for all the words $w \in F(S)$ with $\mu(w) = 1$.

*Remark* 2.3.3. By Corollary 2.2.9, we can formulate an equivalent notion of isoperimetric inequalities by replacing $A(w)$ with $A(\Delta)$, where $\Delta$ is a minimal van Kampen diagram for the word $w$.

This gives rise to a corresponding definition of an isoperimetric inequality for metric spaces.

**Definition 2.3.4** (Isoperimetric inequality for metric spaces)**.** A function $f : [0, \infty) \to [0, \infty)$ is called a *coarse isoperimetric inequality* for a metric space $X$ if there exists $\varepsilon > 0$ such that every rectifiable loop $\gamma$ in X has an $\varepsilon$-filling and $\text{Area}_{\varepsilon}(\gamma) \leq f(l(\gamma))$.

We next define Dehn functions in this context. It is particularly helpful to consider two particular types of metric spaces, which will appear in the next section.

**Definition 2.3.5** (Dehn function, geometric). (i) For $X = \mathrm{Cay}^2(\mathcal{P})$: For an edge-loop $\gamma$ in the Cayley 2-complex of a finite presentation $\mathcal{P}$, $\mathrm{Area}(\gamma)$ is the minimum of $\mathrm{Area}(\Gamma)$ over all van Kampen diagrams spanning $\gamma$.

The Dehn function $\delta_{\mathcal{P}} : \mathbb{N} \to \mathbb{N}$ of a finite presentation $\mathcal{P}$ with Cayley 2-complex $\mathrm{Cay}^2(\mathcal{P})$ is:

$$\delta_{\mathcal{P}}(n) = \max\left\{\mathrm{Area}(\gamma) \mid \text{edge-loops } \gamma \text{ in } \mathrm{Cay}^2(\mathcal{P}) \text{ such that } \ell(\gamma) \le n\right\}.$$

(ii) For $X = \mathcal{C}(G,S)$ where $G = \langle S \mid R\rangle$ is a finitely-presented group:

The Dehn function $\delta_N : \mathbb{N} \to \mathbb{N}$ of a finite presentation with Cayley graph $\mathcal{C}(G,S)$ is:

$$\delta_N(n) = \sup\{\mathrm{Area}_N(\gamma) : \ell(\gamma) \le n\}.$$

It turns out that we can further extend this notion of an isoperimetric inequality to metric spaces using $\varepsilon$-fillings, and these notions of isoperimetry are equivalent.

**Proposition 2.3.6.** *Let $X = \mathcal{C}(G,S)$ where $G = \langle S \mid R\rangle$ is a finitely-presented group. Then, if $N \in \mathbb{N}$ is sufficiently large, the function $\delta_N$ is Dehn equivalent to the Dehn function of the presentation $G = \langle S \mid R\rangle$.*

***Proof.*** For brevity, we leave this as outside the scope of the project, and refer the reader to [4, Exercise (5)]. □

Taking all of this into account, we have started to uncover an equivalence between the algebraic and geometric notions of area and isoperimetry. As we move toward the next section where we demonstrate that a linear Dehn function characterises hyperbolic groups, we will see that the ability to translate between these algebraic and geometric contexts is a useful tool.

# 3 Hyperbolic Groups

## 3.1 Subquadratic implies hyperbolic

The goal for this subsection is to prove Theorem **A**; the existence of a linear Dehn function for some finitely-presented group $G$ implies that the group $G$ is hyperbolic. To do this, we follow and build upon the method outlined in Papasoglu's paper [5]. In particular, we focus on triangular group presentations, which we can use to form a lower bound for $A(w)$.

To start, we recap the definition of a hyperbolic group.

**Definition 3.1.1** (Hyperbolic group)**.** A finitely-generated group $G = \langle S \mid R \rangle$ is *hyperbolic* if its Cayley graph $\mathcal{C} = \mathcal{C}(G, S)$ is hyperbolic as a metric space.

**Definition 3.1.2** (Triangular presentation)**.** A presentation $\langle S \mid R \rangle$ is called *triangular* if all relators $r \in R$ satisfy $\ell(r) = 3$, in other words, all relators have length exactly 3.

**Lemma 3.1.3.** *Let $G$ be a finitely-presented group. Then there exists a triangular presentation of $G$.*

***Proof.*** Let the presentation of $G$ be denoted $\mathcal{P}_1 = \langle S_1 \mid R_1 \rangle$. Consider the presentation $\mathcal{P}_2 = \langle S_2 \mid R_2$ given by taking the set of generators to be the underlying set of elements of $G$, and taking the set of relations to be exactly the products of the form $g_1 g_2 h^{-1}$ where $g_1, g_2 \in G$ and $h$ is the unique element of $G$ such that $h = g_1 g_2$. To show that this really is a presentation of $G$, we argue that we can achieve the presentation $\mathcal{P}_2$ by applying a sequence of *Tietze transformations* to $\mathcal{P}_1$. For more details on Tietze transformations, as well as showing that they send a group presentation to a presentation of an isomorphic group, see [10, p.89–90].

To start, adding all the generators from $\mathcal{P}_2$ to $\mathcal{P}_1$ is a sequence of Tietze transformations, so

$$\langle S_1 \cup S_2 \mid R_1 \rangle = \langle S_1 \mid R_1 \rangle.$$

Since the elements of $S_1$ are contained in the elements of $G$, removing the generators $S_1$ are also Tietze transformations. Therefore,

$$\langle S_2 \mid R_1 \rangle = \langle S_1 \cup S_2 \mid R_1 \rangle = \langle S_1 \mid R_1 \rangle.$$

Moreover, each of the relators $r$ of $R_1$ can be expressed by some product of the form $h = g_1 g_2$ for $g_1, g_2 \in G$. (Otherwise, $r$ is not a relation.) Thus,

$$\langle S_2 \mid R_1 \rangle = \langle S_2 \mid R_1 \rangle = \langle S_1 \cup S_2 \mid R_1 \rangle = \langle S_1 \mid R_1 \rangle.$$

□

Now we know that a triangular presentation exists for any finitely presented group, we proceed to the main result for this subsection.

**Theorem 3.1.4.** *Let $P = \langle S \mid R \rangle$ be a triangular presentation of a group $G$ and suppose that there is an integer $B$ such that every $w \in \langle\langle R \rangle\rangle$ with area $A(w) > B$ satisfies $A(w) < \frac{1}{128}\left(\frac{1}{256}\ell(w)^2 - 3\ell(w)\right)$. Then $G$ is a hyperbolic group.*

***Proof.*** We prove the contrapositive. Assume that $G$ is not a hyperbolic group; it suffices to show that for all non-negative integers $B$, there is some word $w \in \langle\langle R \rangle\rangle$ with $A(w) > B$ such that $A(w) \geq \frac{1}{128}\left(\frac{1}{256}\ell(w)^2 - 3\ell(w)\right)$. Throughout the proof, let $A = \max\{200, B\}$. We will show that $A(w) > A \geq B$, which shows in turn that $A(w) > B$ as required.

**Part 1:**

Here we define a geodesic between points $x, y$ in a metric space $X$ as an isometric map from the interval $[0, d(x, y)]$ to $X$, and take $X$ to be the Cayley graph $\mathcal{C}$ of $G$. (Note that $\mathcal{C}$ is a geodesic metric space.)

We define for $n \in \mathbb{N}$, a function $f_A : \mathbb{N}_{\geq 0} \to \mathbb{N}_{\geq 0}$ by taking the infimum over all geodesics $\gamma, \gamma'$ and all integers $N \in \mathbb{N}_{\geq 0}$ as follows:

$$f_A(n) = \inf_{\substack{N \in \mathbb{N}_{\geq 0} \\ \gamma, \gamma'}} \left\{ d(\gamma(N+n), \gamma'(N+n)) : d(\gamma(N), \gamma'(N)) \geq 6A, \gamma(0) = \gamma'(0) \text{ vertex in } \mathcal{C} \right\}.$$

We write $f$ as a shorthand for $f_A$. Here, distances are defined with respect to the word metric on the Cayley graph $\mathcal{C}$. By the definition of the word metric, all distances have non-negative integer values. Note that $f$ is therefore an integer-valued function.

Since we assume $G$ is not hyperbolic, geodesics in $\mathcal{C}$ do not diverge. Hence $f$ is bounded above, i.e. there exists $K \in \mathbb{N}_{\geq 0}$ such that $\lim_{n \to \infty} f(n) = K < \infty$.

By the properties of convergence of integer-valued functions, there exists some $k \in \mathbb{N}_{\geq 0}$ such that for all $n > k$, $f(n) = K$. Therefore, there exists $n_0 > 2K^2 + 2K$ such that $f(n_0) = K$. Also, by definition of $f$, for every integer $A$ there exist geodesics $\gamma, \gamma'$ with image in $\mathcal{C}$ and values of $n_0, N_0 \in \mathbb{N}_{\geq 0}$ such that $d(\gamma(N_0), \gamma'(N_0)) \geq 6A$ and $d(\gamma(N_0 + n_0), \gamma'(N_0 + n_0)) = K$. (The values $2K^2 + 2K$ and $6A$ are carefully chosen for the proof of Claim 3.1.5, the statement of which is given below.)

Our motivation for constructing the function $f$ lies in the fact that it guarantees the existence of geodesics $\gamma, \gamma'$ with the above properties for a fixed value of $A$. Thus, from here on, we no longer discuss $f$ but instead the geodesics $\gamma, \gamma'$. These properties allow us to prove the following result, which initially appears a bit arbitrary, but helps set up our construction in **Part 2**.

For an integer choice of $M$, let $S = d(\gamma(M), \gamma')$ and $T = \lfloor \frac{S}{3} \rfloor = \lfloor \frac{d(\gamma(M),\gamma')}{3} \rfloor$. We make the following useful claim.

*Claim* 3.1.5. There exists an integer $M$ which satisfies the following conditions:

(i) The distance $d(\gamma(M), \gamma') > 3A$.

(ii) The interval $[M-T, M+T]$ is contained in the interval $[0, N_0 + n_0]$, where $N_0, n_0$ are as previously defined.

(iii) For every integer $k \in [M-T, M+T]$, we have that $d(\gamma(k), \gamma') < d(\gamma(M), \gamma')$.

The proof of this claim involves some unwieldy algebra of inequalities, which we take as a black box [5, p. 152–154]. We now proceed assuming the existence of $M$.

**Part 2:**

We start by forming a construction of geodesic segments. First, we label four points of interest on the geodesics $\gamma, \gamma'$. These are $a_1, a_2, b_1$ and $b_2$, defined as follows. Let $a_2 \in \text{Img}(\gamma')$ be such that $d(\gamma(M-T), a_2) = d(\gamma(M-T, \gamma')$. Similarly, let $b_2 \in \text{Img}(\gamma')$ be such that $d(\gamma(M+T), a_2) = d(\gamma(M+T, \gamma')$. Let $\alpha_1, \alpha_2$ be geodesic arcs joining $\gamma(M-T)$ to $a_2$ and $\gamma(M+T)$ to $b_2$ respectively.

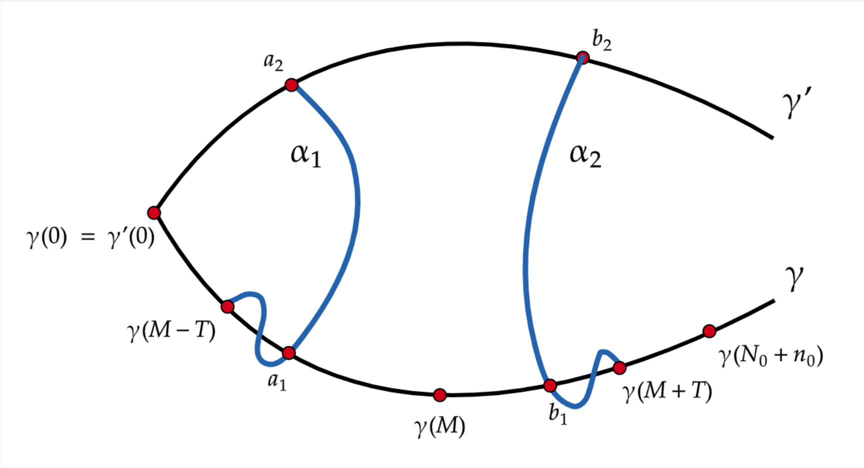


Figure 3: The construction as described.

To define $a_1$ and $b_1$, let

$$t_1 = \sup \{t : \alpha_1(t) \in \text{Img}(\gamma)\}$$
$$t_2 = \sup \{t : \alpha_2(t) \in \text{Img}(\gamma)\}.$$

Then we define $a_1 = \alpha_1(t_1)$ and $b_1 = \alpha_2(t_2)$.

Observe that, by Claim 3.1.5(ii), whenever we have a distance of the form $d(\gamma(s_1), \gamma(s_2))$ for some $s_1, s_2 \in [M-T, M+T]$, this distance is finite and bounded above by $K$. This gives a construction as in Figure 3.

Note that this figure isn't a completely accurate representation of our setup, since the image of the geodesics $\gamma, \gamma'$ lie in the Cayley graph $\mathcal{C}$, however this is still a useful graphic to keep in mind (and much easier to visualise than in the Cayley graph).

Next, we show that $a_1$ is an element of the interval $[\gamma(0), \gamma(M)]$. Suppose not. Then, as $\alpha_1$ is a geodesic path, the closer to $a_1$ that we are along $\gamma$, the closer we are to $a_2$.

Therefore,

$$S = d(\gamma(M), \gamma') \leq d(\gamma(M), a_2) < d(\gamma(M-T), a_2) = d(\gamma(M-T), \gamma').$$

By Claim 3.1.5(iii), we have that

$$d(\gamma(M-T), \gamma') \leq d(\gamma(M), \gamma') = S.$$

Putting the above two lines together, we reach a contradiction. We therefore conclude that $a_1 \in [\gamma(0), \gamma(M)]$, and similarly $b_1 \in [\gamma(M), \gamma(M+T)]$. Additionally, we have that:

$$\begin{aligned}
S &\leq d(\gamma(M), a_2) \\
&\leq d(\gamma(M), a_1) + d(a_1, a_2) && \text{(By triangle inequality)} \\
&= d(\gamma(M), a_1) + d(\gamma(M-T), a_2) - d(\gamma(M-T), a_1) && \text{(As } \alpha_1 \text{ is a geodesic)} \\
&\leq d(\gamma(M), a_1) + M - T + d(\gamma(M), a_1)
\end{aligned}$$

Rearranging this,

$$d(\gamma(M), a_1) \geq \frac{1}{2}T, \text{ and likewise, } d(\gamma(M), b_1) \geq \frac{1}{2}T. \tag{$*$}$$

**Part 3:**

Consider traversing along the rectangle with vertices $a_1, a_2, b_2, b_1$. The edges $[a_1, a_2]$, $[a_2, b_2]$, $[b_2, b_1]$, $[b_1, a_1]$ are paths in the Cayley graph $\mathcal{C}$ and lie on the geodesics $\alpha_1, \gamma', \alpha_2, \gamma$ respectively in $\mathcal{C}$. Since we return to our starting vertex $a_1$, we must have applied a sequence of generators to $a_1$, resulting in a word $w$ equal to the identity in $\langle S \mid R \rangle$. For the rest of the proof, we show that this $w$ is the counter-example we are looking for. In other words, $w$ satisfies $A(w) > A$ and $A(w) \geq \frac{1}{128}\left(\frac{1}{256}\ell(w)^2 - 3\ell(w)\right)$.

By Lemma 2.2.8, there exists a van Kampen diagram with boundary word $w$. Let $\mathcal{D}$ be a minimal such diagram. As the edges are contained in geodesic segments, they are geodesics themselves, and thus

$$\begin{aligned} \ell(w) &= \ell([a_1,a_2]) + \ell([a_2,b_2]) + \ell([b_2,b_1]) + \ell([b_1,a_1]) & (1)\\ &= S + \ell([a_2,b_2]) + S + 2T. & (2) \end{aligned}$$

The second line follows from the fact that $\ell([a_1,a_2]), \ell([b_2,b_1]) < S$ by Claim 3.1.5(iii), and $\ell([b_1,a_1]) < d(\gamma(M-T), \gamma(M+T)) = 2T$.

By the triangle inequality,

$$\begin{aligned} \ell([a_2,b_2]) &\leq \ell([a_2,a_1]) + \ell([a_1,b_1]) + \ell([b_1,b_2]) \\ &\leq 2S + 2T. & (3) \end{aligned}$$

We quickly note that since $\lfloor A \rfloor \leq T$, and $A, T$ are non-negative integers, then $A \leq T$. From this, we may conclude that

$$\begin{aligned} A \leq T &= \frac{1}{2}T + \frac{1}{2}T && \left(\text{By Claim 3.1.5(i) and } T = \frac{S}{3}\right) \\ &\leq \ell([a_1,b_1]) && (\text{By } (*)) \\ &\leq \ell(w) && (\text{By } (1)) \\ &\leq 4S + 4T = 4S + 4\left\lfloor \frac{S}{3} \right\rfloor. && (\text{By (2) and (3)}) \end{aligned}$$

**Part 4:**

We now inductively construct a family of subcomplexes of $\mathcal{D}$. First, some notation: if $\mathcal{S}$ is a subcomplex of $\mathcal{D}$, we denote by $\mathrm{Star}_{\mathcal{D}}(\mathcal{S})$ the set of all closed cells of $\mathcal{D}$ which intersect $\mathcal{S}$.

Let $\mathcal{N}_1 = \mathrm{Star}_{\mathcal{D}}([a_1,b_1])$ and $\mathcal{N}_i = \mathrm{Star}_{\mathcal{D}}(\mathcal{N}_{i-1})$ for $2 \leq i \leq \left\lfloor \frac{T}{16} \right\rfloor - 1$. We will later see why it suffices to consider $i$ in this range.

From this, we consider a family of curves defined by $\gamma_i = \mathcal{N}_i \cap \mathcal{D}^{(1)} \cap (\overline{\mathcal{D}^{\circ} \setminus \mathcal{N}_i})$, where $\mathcal{D}^{(1)}$ is the 1-skeleton of $\mathcal{D}$. To motivate this, we briefly visualise these curves, shown in green in Figure 4. Note that $\mathcal{D}$ is a van Kampen diagram for a group with triangular presentation, meaning all 2-cells in $\mathcal{D}$ are triangular. Loosely speaking, the curves $\gamma_i$ divide the boundary rectangle of $\mathcal{D}$ into 'layers', where each layer is a chain of triangular 2-cells of height at most 1. The dissection of the area between $\alpha_1$ and $\alpha_2$ into layers will allow us to determine a lower bound for $A(w)$.

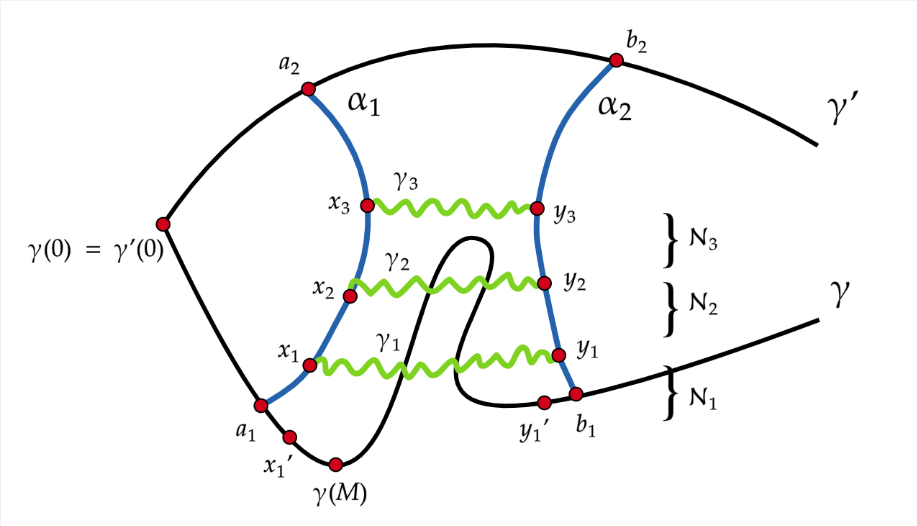


Figure 4: Family of subcomplexes of $\mathcal{D}$.

*Claim* 3.1.6. The length $\ell(\gamma_i) \geq \frac{T}{4}$ for all $1 \leq i \leq \left\lfloor \frac{T}{16} \right\rfloor - 1$. If $\gamma_i$ is not connected, we define $\ell(\gamma_i)$ to be the sum of the lengths of the connected components.

***Proof of Claim 3.1.6.*** Let $\beta_i$ be the connected component of $\gamma_i$ for which $\beta_i \cap [a_1, a_2]$ is a single point. Call this point $x_i$. Similarly, let $\beta_i'$ be the connected component of $\gamma_i$ such that $\beta_i' \cap [b_1, b_2]$ is a single point, call this point $y_i$. (If $\gamma_i$ is connected, we have $\beta_i = \beta_i'$.)

Note that there are points $x_i', y_i' \in [a_1, b_1]$ such that $d(x_i, x_i') = i$, $d(y_i, y_i') = i$. This is because $d(x_1, x_i) = i$, so there is at least one point which satisfies this condition. A useful observation is that the points $x_i'$ and $y_i'$ are on opposite sides of $\gamma(M)$. This is because

$$d(a_1, x_i') \leq d(a_1, x_i) + d(x_i, x_i') = i + i = 2i \leq 2\left\lfloor \frac{T}{16} \right\rfloor - 2 < \frac{1}{2}T, \text{ and}$$
$$d(a_1, \gamma(M)) \geq \frac{1}{2}T \text{ by } (*).$$

Hence, $x_i' \in [a_1, \gamma(M)]$, and similarly $y_i' \in [\gamma(M), b_1]$.

To prove the claim, it now remains to do some careful algebraic manipulation of inequalities. We have

$$\begin{aligned} S &\leq d(\gamma(M), a_2) \\ &\leq d(\gamma(M), x_i') + d(x_i', x_i) + d(x_i, a_2) \\ &= d(\gamma(M), x_i') + d(x_i', x_i) + d(a_1, a_2) - d(a_1, x_i) \end{aligned}$$

$$\leq d(\gamma(M), x_i') + d(x_i', x_i) + S - \big(d(a_1, x_i') - d(x_i', x_i)\big)$$

where the second and third lines result from the triangle inequality and reverse triangle inequality respectively, and the final line uses the definition of $S$.

Therefore,

$$d(\gamma(M), x_i') \geq d(a_1, x_i') - 2i \tag{4}$$

$$d(\gamma(M), x_i') + d(a_1, x_i') \geq \frac{1}{2}T \tag{5}$$

where (4) follows from above, and (5) is given by ($*$) and the triangle inequality. Adding (4) and (5) gives:

$$\begin{aligned} 2d(\gamma(M), x_i') &\geq \frac{1}{2}T - 2i \\ \implies d(\gamma(M), x_i') &\geq \frac{1}{4}T - i \\ \implies d(\gamma(M), x_i') &\geq \frac{1}{4}T - \frac{1}{16}T = \frac{3}{16}T. \end{aligned}$$

Similarly, $d(\gamma(M), y_i') \geq \frac{3}{16}T$. Putting this all together,

$$\begin{aligned} \ell(\gamma_i) \geq d(x_i, y_i) &\geq d(x_i', y_i') - d(x_i, x_i') - d(y_i, y_i') \\ &= d(x_i', \gamma(M)) + d(\gamma(M), y_i') - i - i \\ &\geq \frac{6}{16}T - 2\left(\left\lfloor \frac{T}{16} \right\rfloor - 1\right) \\ &\geq \frac{6}{16}T - 2\left(\frac{T}{16}\right) = \frac{1}{4}T \end{aligned}$$

which proves the claim. □

**Part 5:**

Finally we use this result to make conclusions about the area of the boundary word $w$. First consider the layer $\mathcal{N}_{i+1} \setminus \mathcal{N}_i$ and notice that each closed triangular 2-cell of $\mathcal{N}_{i+1} \setminus \mathcal{N}_i$ has at most two edges on $\gamma_i$. As each cell has height at most 1, it follows that

$$\text{Area}(\mathcal{N}_{i+1} \setminus \mathcal{N}_i) \geq \frac{\ell(\gamma_i)}{2} \leq \frac{1}{8}T.$$

From here, we make two observations. Firstly,

$$A(w) \geq \sum_{i=1}^{\lfloor \frac{T}{16} \rfloor - 2} \text{Area}(\mathcal{N}_{i+1} \setminus \mathcal{N}_i) \geq \left(\left\lfloor \frac{T}{16} \right\rfloor - 2\right) \cdot \frac{1}{8}T = \frac{1}{8}T\left(\frac{T}{16} - 3\right).$$

Secondly, by **Part 3**: $\ell(w) \leq 4S + 4T \leq 12\left\lfloor \frac{S}{3} \right\rfloor + 12 + 4T = 16T + 12$.

Combining these results, we may deduce that

$$A(w) \geq \frac{1}{128}\left(T^2 - 48T\right) \geq \frac{1}{128}\left(\frac{1}{256}\ell(w)^2 - \frac{99}{32}\ell(w)\right) > \frac{1}{128}\left(\frac{1}{256}\ell(w)^2 - 3\ell(w)\right)$$

as required.

It remains to check that $A(w) > A$, where $A = \max\{200, B\}$. It is quick to check that $A(w) \geq \frac{1}{128}(T^2 - 48T) > A$ if and only if $T > 24 + \sqrt{576 + 128A}$. Since $T \geq \lfloor A \rfloor > A - 1$, if $A - 1 > 24 + \sqrt{576 + 128A}$ we have the desired result. Solving this inequality as a quadratic in $A$, this holds for all $A > 89 + 8\sqrt{123} \approx 178$. By definition of $A = \max\{200, B\}$ this condition is satisfied, and hence $A(w) > A \geq B$. □

We note that the above proof hinges on a very specific bound for the area of $w$, but this suffices for all subquadratic Dehn functions by Dehn equivalence. Therefore, we have shown that for any finitely presented group $G$, if $G$ has a subquadratic Dehn function, then $G$ is hyperbolic.

## 3.2 Hyperbolic implies linear

To assemble the proof of the converse implication, we require some background knowledge on metric graphs, which we combine with our background on area fillings of loops and notes in the appendix concerning local geodesics. We define each of these and explore some of their useful properties in the proceeding subsections, using [4, Chapter III.H].

**Lemma 3.2.1.** *Let $X$ be a metric graph whose edges all have integer lengths, and suppose that $X$ is $\delta$-hyperbolic where $\delta > 0$ is an integer. Given any non-trivial rectifiable loop $\gamma : [a, b] \to X$ which begins (and ends) at a vertex, one can find $s, t \in [a, b]$ such that*

*(i) $\gamma(s)$ and $\gamma(t)$ are vertices of $X$,*

*(ii) $d(\gamma(s), \gamma(t)) \leq l(\gamma_{[s,t]}) - 1$, and*

*(iii) $d(\gamma(s), \gamma(t)) + l(\gamma_{[s,t]}) \leq 16\delta$.*

***Proof.*** We claim that $X$ contains no loops which are $k$-local geodesics for $k = 8\delta + \frac{1}{2}$. Suppose, for a contradiction, that an $(8\delta + \frac{1}{2})$-local loop $\gamma_0 : [0, 1] \to X$ does exist. (Since $\gamma_0$ is a loop, we have that $\gamma_0(0) = \gamma_0(1)$). Since $X$ is a $\delta$-hyperbolic geodesic metric space, by Theorem A.1.2, we have that $\mathrm{Img}(\gamma_0) \subset B_{2\delta}(\gamma_0(0))$. It suffices to show that there exists some time $t$ on which $\gamma_0$ is defined such that $d(\gamma_0(0), \gamma_0(t)) > 2\delta$, in other words, there is some time at which the loop exits the ball. Consider $t = 8\delta$. Since $\gamma_0$ is $(8\delta + \frac{1}{2})$-local, $d(\gamma_0(0), \gamma_0(t)) = 8\delta > 2\delta$ as required.

Thus, by the above claim, there exists a non-geodesic subarc $\gamma\restriction_{[s_0,t_0]}$ of $\gamma$ with path length less than $8\delta + \frac{1}{2}$. Next, choose a geodesic connecting $\gamma(s_0)$ and $\gamma(t_0)$. (This exists since $X$ is a geodesic metric space). Call this geodesic $\sigma : [0,1] \to X$. Note that these two points do not necessarily have to vertices, but could be points partway along an edge. We define $\gamma(s)$ and $\gamma(t)$ to be the first and last vertices of $X$ respectively through which this geodesic passes. This shows (i) constructively.

For (ii), note that for any two rectifiable paths in $X$ with common endpoints, the difference in the lengths of these paths is an integer. Applying this to the paths $\gamma\restriction_{[s_0,t_0]}$ and $\sigma$, this gives us that $l(\gamma_{[s,t]}) - d(\gamma(s),\gamma(t)) \geq 1$, which rearranges to give (ii).

Finally, (iii) follows as a result of (ii) as follows:

$$\begin{aligned} d(\gamma(s),\gamma(t)) + l(\gamma_{[s,t]}) &\leq 2 \cdot l(\gamma_{[s,t]}) - 1 \\ &< 2\left(8\delta + \frac{1}{2}\right) - 1 = 16\delta. \end{aligned}$$

□

The above result is a useful ingredient in the proof of our main theorem of this section, which we discuss next. This method of filling edge-loops is taken from [4, p. 417–418], the earliest application of which was due to Max Dehn [11] in a foundational work published in 1912.

**Theorem 3.2.2.** *Let $X$ be a geodesic space. If $X$ is $\delta$-hyperbolic, then it satisfies a linear isoperimetric inequality.*

***Proof.*** We start by combining several of the previous results in order to simplify our end goal. By Proposition A.2.1, a $\delta$-hyperbolic metric space $X$ is quasi-isometric to a metric graph $X'$ with unit edge lengths. As quasi-isometric embeddings preserve hyperbolicity (Theorem A.1.3), this implies that the graph $X'$ is $\delta$-hyperbolic for some $\delta > 0$. If we can show that $X'$ satisfies a linear isoperimetric inequality, then it will follow from Proposition A.2.2 that $X$ satisfies a linear isoperimetric inequality. Hence, it suffices to show that the metric graph $X'$ satisfies a linear isoperimetric inequality.

In order to avoid writing $X'$ repetitively instead of $X$, we can assume that $X$ is a metric graph with unit edge lengths. Assume that $X$ is $\delta$-hyperbolic where $\delta$ is an integer (i.e. $\delta \geq 1$). To outline the proof, we aim to show by induction on $l(\gamma) + l_0(\gamma)$ that every edge-loop in $X$ admits a standard $16\delta$-filling of area $(8\delta + 2)(l(\gamma) + l_0(\gamma))$. From this, it will then follow that $X$ satisfies a linear isoperimetric inequality.

We start by considering the first case of the induction. Consider the trivial case in which $\gamma$ is defined as an edge-loop given by a constant map at a single vertex. Here, $l(\gamma) = 0$ and $l_0(\gamma) = 1$. Since there are no faces of $P$, trivially there exists an $\varepsilon$-filling of $\gamma$ for any value of $\varepsilon \geq 0$. In addition, area of the filling corresponds to the number of faces of $P$, which is 0. In this case $\gamma$ admits a standard $16\delta$-filling of area $0 < (8\delta + 2)(l(\gamma) + l_0(\gamma)) = 8\delta + 2$.

For the inductive step, suppose our proposition holds for $n$ up to and including $n = l(\gamma) + l_0(\gamma) - 1$. The goal of our inductive argument is therefore to show it holds for $n = l(\gamma) + l_0(\gamma)$. Given an edge-loop $\gamma : \mathbb{S}^1 \to X$ with $l(\gamma) \geq 2$, we consider how to reduce $l(\gamma) + l_0(\gamma)$. If $l_0(\gamma) = 0$, then:

*Case 1.* The path $\gamma$ is locally injective, or;

*Case 2.* The path $\gamma$ contains a subpath which backtracks. In other words, there is a subpath which traverses a sequence of edges and then immediately returns along those edges.

(To see this, note that if neither case 1 nor case 2 hold, then there exists some interval $[t_1, t_2]$ such that $\gamma([t_1, t_2])$ is a constant map at a vertex.)

We construct a filling for $\gamma$ in each of the above two cases. For case 1, choose $s$ and $t$ as in Lemma 3.2.1 and map the interval segment $[s, t]$ to a constant speed parameterisation of a geodesic segment joining $c(s)$ to $c(t)$ in $X$. For case 2, connect the endpoints of the backtracking subpath (we can think of this as the times at which we start and stop traversing this backtracking subpath) by a Euclidean segment in the disk. We can "remove" this backtracking segment by sending this segment to $X$ by a constant map.

Now we consider the case where $l_0(\gamma) \geq 1$. For each maximal subpath of $\gamma$ which a constant map at a vertex, we can consider a subpath $\gamma \restriction_{[s,t]}$ which is the concatenation of this constant map with the proceeding subpath of length one. We again connect $s$ to $t$ by a Euclidean segment in the disk and map this segment to $X$ as a constant speed parametrisation of a geodesic segment $[\gamma(s), \gamma(t)]$, thereby "removing" the subpath of $\gamma$ which is a constant map.

In each of these three situations, we have started to fill the edge-loop $\gamma$ by dividing the disk into two sectors. The "small" sector has a boundary which maps to an edge-loop (containing $\gamma([s, t])$) of length at most $16\delta$ by Lemma 3.2.1. The other "big" sector has boundary map which is an edge-loop $\gamma'$ with $l_0(\gamma') + l(\gamma') < l_0(\gamma) + l(\gamma)$. By our assumption, we may fill this big sector with a standard $16\delta$-filling $(P, \Phi)$ of $\gamma'$ with $\text{Area}_{16\delta}(\gamma') \leq (8\delta + 2)(l_0(\gamma') + l(\gamma') - 1)$. We may assume that $s$ and $t$ are vertices of $P$, since if not, we may subdivide to create two additional faces such that this is the case, as shown in Figure 5.

Now we consider how to fill the small sector. In order to do this, we use that the restriction of $\Phi$ to the Euclidean segment $[s,t]$ is a concatenation of at most $8\delta$ edges. This follows as a consequence of Theorem A.1.2. By this observation, it then follows that the interior of the segment $[s,t]$ contains fewer than $8\delta$ vertices from the triangulation of the filling. To complete the standard filling of $\gamma$, we introduce edges connecting this set of vertices to a vertex introduced on $\mathbb{S}^1$ between $s$ and $t$.

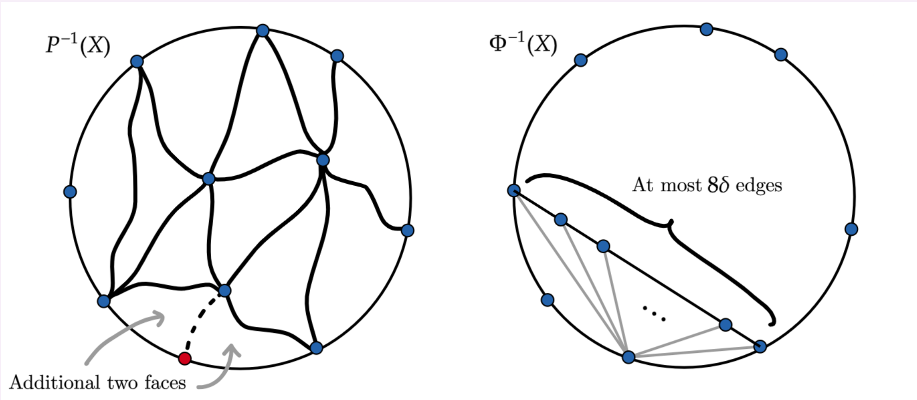


Figure 5: On the left, an example triangulation $P$ showing the creation of two faces (+1 to the total number of faces) when the red vertex corresponding to $s$ or $t$ is added. On the right, a visualisation of how the small sector is filled. (The other vertices interior to the disk are removed for clarity.)

These $8\delta$ faces, plus two faces from subdividing so that $s,t$ are vertices of $P$ gives us at most an additional $8\delta+2$ faces. Hence for $n = l(\gamma)+l_0(\gamma)$, we have shown inductively that every edge-loop in $X$ admits a standard $16\delta$-filling of area $(8\delta+2)(l(\gamma)+l_0(\gamma))$.

It remains to justify that $X$ satisfies a linear isoperimetric inequality. Since our construction bounds $\text{Area}_{16\delta}(\gamma)$ by a linear function of $l(\gamma)$, we have by Definitions 2.3.4 and 2.3.5 that $X$ satisfies a linear isoperimetric inequality. □

To conclude this section, we recall Theorem **A** and **B** from the end of Section 2.1, and relate these to Theorem 3.1.4 (subquadratic Dehn function implies hyperbolic group) and Theorem 3.2.2 as above. Together, this gives us our desired result that a group with subquadratic Dehn function in fact admits a linear Dehn function. Therefore, we have shown that the Gromov gap corresponding to the interval $(1,2)$ exists.

In the next section, we introduce topological constructions in order to prove that this gap is the only gap in the isoperimetric spectrum.

# 4 Snowflake Groups

In this section, we show that there is only one gap in the isoperimetric spectrum. Since we refer to this result frequently, we refer to it by Theorem **D** as given below.

**Theorem D.** *[6, p. 2] The closure of* $\mathbb{P}$ *is* $1 \cup [2, \infty)$.

In order to demonstrate this, we prove a stronger result which will in turn imply Theorem **D**.

**Theorem 4.0.1.** *[6, p. 2] For all pairs of positive integers* $p > q$*, there exists a finitely presented group* $G$ *whose Dehn function* $\delta_G(n) \simeq n^{2\alpha}$ *where* $\alpha = \log_2\left(\frac{2p}{q}\right)$.

We prove Theorem 4.0.1 by introducing a construction for a specific family of groups, termed *snowflake groups*, and determining a lower and upper bound for the Dehn function of a particular presentation of these groups. This approach aligns with the existing literature, where these groups have been previously studied by Brady [3, Chapter I.1] and others ([6], [12]). In this section, we follow the exposition in Brady's text in [3, p. 10–25].

Before defining snowflake groups formally, we introduce two key topological constructions: graphs of spaces and graphs of groups, along with the torus construction. Since snowflake groups are defined as graphs of groups, we start by establishing these definitions.

## 4.1 Graphs of groups

The simplest way to define snowflake groups is using a construction known as an *HNN extension* (named after Higman, Neumann and Neumann, 1949). This is a way to extend a group $G$ by adjoining a new element under certain conditions, described formally as follows.

**Definition 4.1.1** (HNN extension)**.** Let $G$ be a group with presentation $\langle S \mid R \rangle$, and consider $A, B$ subgroups of $G$. Let $\rho : A \to B$ an isomorphism between two subgroups, and let $t \notin G$ be a new element such that $\langle t \rangle$ is the cyclic group of infinite order. Then, the *HNN extension* of $G$ relative to $A, B$ and $\rho$ is:

$$G*_\rho = \langle S, t \mid R, t^{-1}at = \rho(a) \text{ for all } a \in A \rangle.$$

The new generator $t$ is often referred to as a *stable letter.*

Using HNN extensions allows us to invoke a useful result known as Britton's lemma.

**Definition 4.1.2.** Let $G*_\rho$ be a HNN extension using the same notation as above. For $n \in \mathbb{N}$, let $\{g_i\}_{i=1}^n$ be a sequence of words, each expressed in the generating set $S \subset G$, and let the powers $\{a_i\}_{i=1}^n$ denote either $\pm 1$. A sequence $g_0, t^{a_1}, g_1, t^{a_2}, g_2, \ldots, t^{a_n}, g_n$ is called *reduced* if there is no consecutive subsequence of the form:

1. $t^{-1}, g_i, t$ with $g_i \in A$, or;

2. $t, g_j, t^{-1}$ with $g_j \in B$.

**Lemma 4.1.3** (Britton's Lemma)**.** *Let $n \geq 1$. If the sequence $g_0, t^{a_1}, g_1, t^{a_2}, g_2, \ldots, t^{a_n}, g_n$ is reduced, then $g_0 t^{a_1} g_1 t^{a_2} g_2 \cdots t^{a_n} g_n \neq 1$ in $G{\star}_\rho$.*

We refer the interested reader to [10, p. 181] for a proof.

**Definition 4.1.4** (Snowflake groups)**.** The *snowflake groups* are groups given by group presentations of the form

$$G_{p,q} = \left\langle a, b, s, t \,\middle|\, [a,b] = 1, sa^q s^{-1} = a^p b, ta^q t^{-1} = a^p b^{-1} \right\rangle.$$

We can think of this as the multiple HNN extension of $G = \langle a, b \mid [a,b] = 1 \rangle \cong \mathbb{Z}^2$ relative to the isomorphisms $\rho_1 : \langle a^q \rangle \to \langle a^p b \rangle$ and $\rho_2 : \langle a^q \rangle \to \langle a^p b^{-1} \rangle$. (These are isomorphic since each of these cyclic groups are proper non-trivial subgroups of $\mathbb{Z}^2$.)

We recall the definition of a (directed) graph from *MA241 Combinatorics* and similar modules. The following definitions are taken from [3, p. 10].

**Definition 4.1.5** (Initial, terminal vertex)**.** Let $\Gamma = (\mathcal{V}(\Gamma), \mathcal{E}(\Gamma))$ be a directed graph. By definition, each $e \in \mathcal{E}(\Gamma)$ is an ordered pair of vertices $(v_1, v_2)$ for $v_1, v_2 \in \mathcal{V}(\Gamma)$. Let the maps $\iota$ and $\tau$ send each edge to one of the two incident vertices as follows:

$$\begin{aligned} \iota : \mathcal{E}(\Gamma) &\to \mathcal{V}(\Gamma) \\ e = (v_1, v_2) &\mapsto v_1 \end{aligned} \qquad\qquad \begin{aligned} \tau : \mathcal{E}(\Gamma) &\to \mathcal{V}(\Gamma) \\ e = (v_1, v_2) &\mapsto v_2 \end{aligned}$$

For each edge $e \in \mathcal{E}(\Gamma)$, we call $\iota(e)$ the *initial vertex* and $\tau(e)$ the *terminal vertex.*

**Definition 4.1.6** (Graph of spaces)**.** [2, p. 10] A *graph of spaces* is a finite directed graph $\Gamma$ together with the following data:

1. A *vertex space* $X_v$ associated to each vertex $v \in \mathcal{V}(\Gamma)$,
2. An *edge space* $X_e$ associated to each edge $e \in \mathcal{E}(\Gamma)$, and
3. A collection of continuous maps $f_{\iota,e} : X_e \to X_{\iota(e)}$ and $f_{\tau,e} : X_e \to X_{\tau(e)}$ for each edge $e \in \mathcal{E}(\Gamma)$.

**Definition 4.1.7** (Total space)**.** The *total space* of a graph of spaces is defined as the quotient space of

$$\left( \bigcup_{v \in V(\Gamma)} X_v \right) \cup \left( \bigcup_{e \in E(\Gamma)} X_e \times [0,1] \right)$$

by the identifications $(x, 0) \sim f_{\iota(e)}(x)$ for all $x \in X_e$ and $(x, 1) \sim f_{\tau(e)}(x)$ for all $x \in X_e$.

**Definition 4.1.8** (Fundamental group of graph of spaces)**.** The *fundamental group of a graph of spaces* is the fundamental group of the total space.

Given a group $G$, one can consider the presentation 2-complex (recall Definition 2.2.2) corresponding to a given presentation of $G$.

**Definition 4.1.9** (Aspherical)**.** For a group $G$ with presentation $\langle S \mid R \rangle$, a presentation 2-complex $K_{\langle S|R \rangle}$ is said to be *aspherical* if its universal covering space is contractible.

We now define what it means to be a graph of groups, which is a specific case of the more general graph of spaces.

**Definition 4.1.10** (Graph of groups)**.** A *graph of groups* is a finite directed graph $\Gamma$ together with the following data:

1. A *vertex group* $G_v$ associated to each vertex $v \in \mathcal{V}(\Gamma)$,
2. An *edge group* $G_e$ associated to each edge $e \in \mathcal{E}(\Gamma)$, and
3. A collection of injective homomorphisms $\varphi_{\iota,e} : G_e \to G_{\iota(e)}$ and $\varphi_{\tau,e} : G_e \to G_{\tau(e)}$ for each edge $e \in \mathcal{E}(\Gamma)$.

**Proposition 4.1.11.** *The snowflake groups are graphs of groups, and the total space for the corresponding graph of spaces is aspherical.*

***Proof.*** For each snowflake group $G_{p,q}$, this is in fact a graph of groups where the underlying graph is a bouquet of two circles. The vertex group is $\mathbb{Z}^2$, generated by $a$ and $b$, and the edge groups are both $\mathbb{Z}$.

Pictorially, we can visualise the total space as obtained by attaching two cylinders to the torus. To do this, one end of each of the two cylinders is attached to the torus along the curve $a^q$ (i.e. $q$ times around the loop $a$). The other end of the first cylinder is identified with $a^p b$ and the other end of the second cylinder is identified with $a^p b^{-1}$. This construction is shown in Figure 6 below.

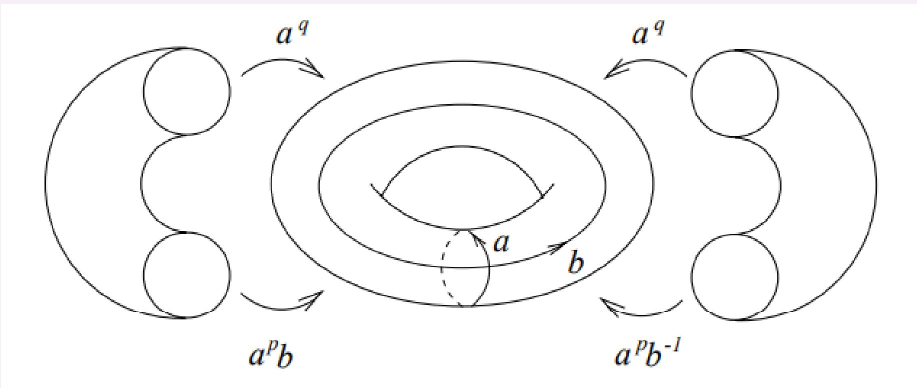


Figure 6: The 2-complex for $G_{p,q}$. This figure is taken directly from [6, p. 1057].

The universal cover of this total space is a collection of planes (covers of the torus) indexed by the cosets of $\langle a, b \mid [a, b] \rangle$ in $G_{p,q}$, glued along strips (covers of the cylinders). This is a contractible 2-complex, which means that the total space is aspherical. □

## 4.2 Lower bound

After establishing some background on snowflake groups, we move towards determining a lower bound for the Dehn function $\delta_{G_{p,q}}$. In the next subsection, we will show that we can find an upper bound of the same order of $\ell(w)$, thereby giving us the Dehn function $\delta_{G_{p,q}}$. Before we begin we assume the following result.

**Proposition 4.2.1.** *If a van Kampen diagram $\Delta$ embeds into the universal cover $\tilde{K}$ for some aspherical 2-complex $K$, then any van Kampen diagram $\Delta'$ with the same boundary label as $\partial\Delta$ has at least as many 2-cells as $\Delta$. In other words,* $\mathrm{A}(\Delta) \leq \mathrm{A}(\Delta')$.

***Proof.*** An explanation is given in [6, p. 1057]. □

**Theorem 4.2.2** (Lower bound for Dehn function)**.** *The Dehn function $\delta_{G_{p,q}}$ for the snowflake groups satisfies the inequality $n^{2\alpha} \leq \delta_{G_{p,q}}(n)$, where $\alpha = \log_2(\frac{2p}{q})$.*

***Proof.*** In the proof of Proposition 4.1.11, we described the universal cover of the total space of the graph of spaces by a collection of copies of $\mathbb{R}^2$ (universal covers of the torus) connected together in a tree-like manner by strips (universal covers of the cylinders).

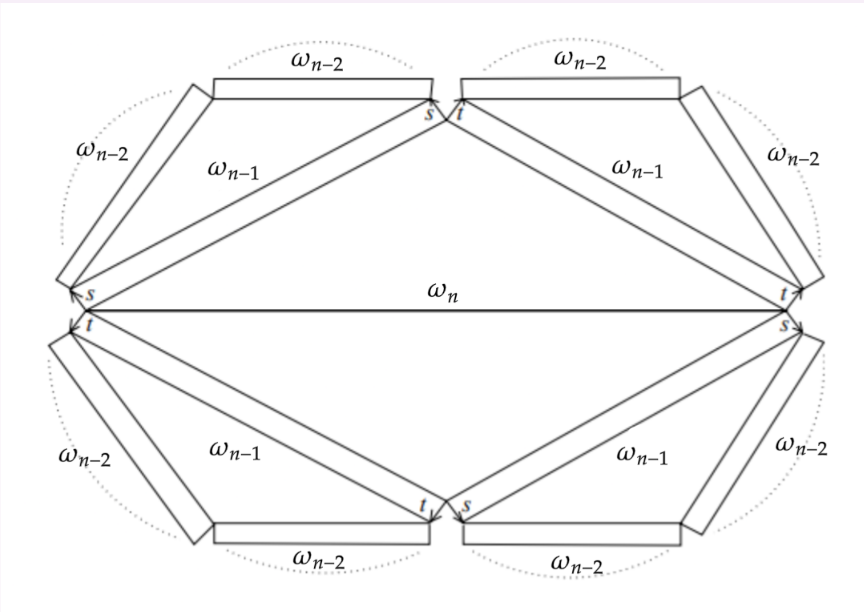


Figure 7: The iterated snowflake diagram, modified from Brady and Bridson's paper [6, p. 1058].

To prove this, we follow the proof in Brady and Bridson's paper [6, p. 1058–9] to inductively construct an embedded disk in the universal cover. From this, we establish a sequence of null-homotopic words $W_k$ in the generators $\{a, s, t\}$ of length $l_k$ which have area of order $l_k^{2\alpha}$, where $\alpha = \log_2(\frac{2p}{q})$ as in the theorem statement. We refer to these as *snowflake words*. We keep in mind the *snowflake diagram* as pictured in Figure 7. We will express the snowflake words $W_k$ in terms of words $w_k$ which concern the upper half of the boundary of this picture. (Geometrically, the lower half of the diagram will look like a reflected image of the upper half, and only the labels will be different. Therefore, we can restrict our attention to the upper half of the diagram.)

To start, let $w_0 := a^q$ and $w_1 := sa^q a^{-1} ta^q t^{-1}$. Notice that

$$w_1 \equiv (sa^q a^{-1})(ta^q t^{-1}) \equiv a^p ba^p b^{-1} \underset{G_{p,q}}{=} a^{2p}.$$

Next, we define

$$\begin{aligned} w_2 &:= sw_1 a^{\varepsilon_1} s^{-1} tw_1 a^{\varepsilon_1} t^{-1} \\ &\equiv sa^{2p+\varepsilon_1} s^{-1} ta^{2p+\varepsilon_1} t^{-1} \end{aligned}$$

where $\varepsilon_1$ is the smallest positive integer so that $q$ divides $2p + \varepsilon_1$. Continuing in this way, we define

$$w_i := sw_{i-1} a^{\varepsilon_{i-1}} s^{-1} tw_{i-1} a^{\varepsilon_{i-1}} t^{-1}$$

where $\varepsilon_{i-1}$ is chosen so that $0 \leq \varepsilon_{i-1} \leq q - 1$ and $w_{i-1} a^{\varepsilon_{i-1}} \underset{G_{p,q}}{=} a^m$ for some exponent such that $m$ is divisible by $q$.

After $k$ iterations, we reach a word $w_k$ which satisfies $4(2^k) \leq \ell(w_k) \leq (4q)2^k$ such that $w_k \underset{G_{p,q}}{=} a^{m_k}$, where

$$m_k = \left(\frac{2p}{q}\right)^k q + \varepsilon_1 \left(\frac{2p}{q}\right)^{k-1} + \cdots + \varepsilon_{k-1} \left(\frac{2p}{q}\right).$$

For us, it suffices that $m_k \geq \left(\frac{2p}{q}\right)^k q$. We note this observation and return to it shortly. In addition, we observe that the van Kampen diagram representing the equality $w_k = a^{m_k}$ is embedded in the universal cover for the total space given by the graph of groups corresponding to the snowflake groups $G_{p,q}$.

We claim that the snowflake words are the given by the commutators

$$W_k = [sw_{k-1} a^{\varepsilon_{k-1}} s^{-1}, tw_{k-1} a^{\varepsilon_{k-1}} t^{-1}].$$

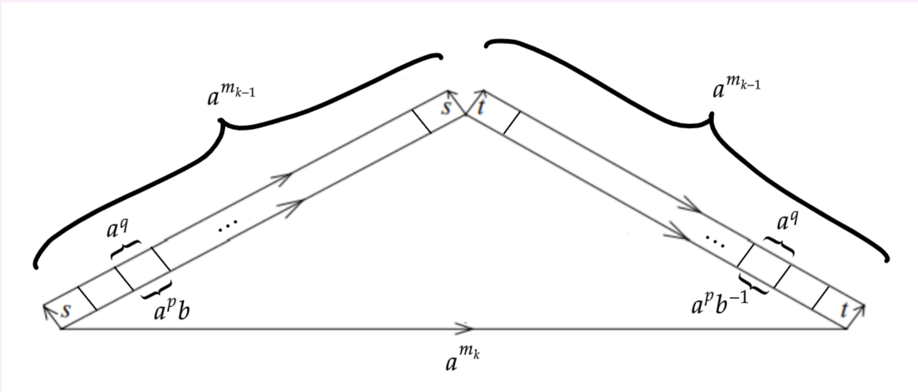


Figure 8: Close-up diagram displaying the relation $w_k = a^{m_k}$, also modified from Brady and Bridson's paper [6, p. 1059].

We have that

$$\begin{aligned} W_k &= [sw_{k-1}a^{\varepsilon_{k-1}}s^{-1}, tw_{k-1}a^{\varepsilon_{k-1}t^{-1}}] \\ &\equiv sw_{k-1}a^{\varepsilon_{k-1}}s^{-1}tw_{k-1}a^{\varepsilon_{k-1}t^{-1}}(sw_{k-1}a^{\varepsilon_{k-1}}s^{-1})^{-1}(tw_{k-1}a^{\varepsilon_{k-1}t^{-1}})^{-1} \\ &\equiv w_k w_k^{-1} \underset{G_{p,q}}{=} 1 \end{aligned}$$

and that the length $\ell(W_k) = 2\ell(w_k)$. By Figure 8, this word is the boundary of an embedded van Kampen diagram (which we refer to as the snowflake diagram) in the universal cover. The central square in this snowflake diagram is a van Kampen diagram over the subpresentation $\langle a, b \mid [a, b] \rangle$ with diagonal labelled $a^{m_k}$. Therefore, by Proposition 4.2.1, there exists some constant $C > 0$ such that the area of this square subdiagram is at least $Cm_k^2$. Hence, it follows that

$$\mathrm{A}(W_k) \geq Cm_k^2 \geq Cq^2 \left(\frac{2p}{q}\right)^{2k}.$$

Additionally, by our constraints on $\ell(w_k)$ determined previously,

$$8(2^k) \leq \ell(W_k) = 2\ell(w_k) \leq (4q)2^{k+1}.$$

Putting this all together,

$$\begin{aligned} \ell(W_k)^{2\log_2\left(\frac{2p}{q}\right)} &= (2 \cdot (4q)2^k)^{2\log_2\left(\frac{2p}{q}\right)} \\ &= (8q)^{2\log_2\left(\frac{2p}{q}\right)} \cdot (2^k)^{2\log_2\left(\frac{2p}{q}\right)} \\ &\leq (8q)^{2\log_2\left(\frac{2p}{q}\right)} \cdot \left(\frac{2p}{q}\right)^{2k} \leq A(W_k). \end{aligned}$$

□

## 4.3 Upper bound

In this subsection, we continue to closely follow the work by Brady and Bridson in [6] to establish an upper bound for the Dehn function $\delta_{G_{p,q}}$. For brevity, we will rely on several key results presented in [6] without explicit proof. However, we demonstrate how these results combine to culminate in the final proof. This will hopefully provide a high-level overview of the underlying approach.

**Definition 4.3.1** (Torus subgroup)**.** We refer to the subgroup $\langle a, b \mid [a, b] = 1\rangle \subset G_{p,q}$ as the *torus subgroup* of the snowflake group.

To determine an upper bound for the Dehn function, we introduce a useful idea known as distortion. Loosely speaking, this is a measure of the reduction in length of a word $w$ when extra generators are introduced.

**Definition 4.3.2** (Distortion)**.** For a pair of groups $H \subset G$ with finite generating sets, let the word metrics be $d_H$ and $d_G$ of $H$ and $G$ respectively. The *distortion of $H$ in $G$* is the function $d : \mathbb{N} \to \mathbb{N}$ defined by

$$d(n) = \max\{d_H(1, h) \mid h \in H,\ d_G(1, h) \le n\}.$$

A useful tool for the proof is the Hölder inequality for sums, stated without proof as follows.

**Proposition 4.3.3** (Hölder inequality)**.** *For constants $\alpha \ge 1$ and $x_i > 0$, we have that:*

$$\left(\sum_{i=1}^{n} x_i\right)^{\alpha} \ge \left(\sum_{i=1}^{n} x_i^{\alpha}\right).$$

**Definition 4.3.4** (Inverse pair)**.** Suppose that $w = x_1 x_2 \dots x_l$ represents a word in the subgroup $\langle a, b \mid [a, b] = 1\rangle \subset G_{p,q}$ and the $x_i$ are one of the six generators $a^{\pm1}, s^{\pm1}, t^{\pm1}$. An *inverse pair* is an ordered pair $(x_i, x_j)$ with $i < j$ such that the following properties hold:

1. Either $i = 1$ or $w = x_1 x_2 \dots x_l \in \langle a, b \mid [a, b] = 1\rangle$.
2. The element $x_1 \dots x_k \notin \langle a, b \mid [a, b] = 1\rangle$ for all $i \le k \le j - 1$.
3. The element $x_1 \dots x_j \in \langle a, b \mid [a, b] = 1\rangle$.

By retracting $G_{p,q}$ onto the free group $F(\{s, t\})$, if $(x_i, x_j)$ is an inverse pair, then $x_i = x_j^{-1} \in \{s^{\pm1}, t^{\pm1}\}$, and the subword determined by an inverse pair represents an element by one of the groups $\langle a^q\rangle$, $\langle a^q b\rangle$, or $\langle a^q b^{-1}\rangle$. Since the inverse pairs of $w$ are non-overlapping nested pairs, we have that any word $w$ which represents a word in the subgroup $\langle a, b \mid [a, b] = 1\rangle \subset G_{p,q}$ has a unique decomposition as a product of subwords of five types:

*Type 1.* These are the subwords of the form $s \dots s^{-1}$ which are determined by the inverse pairs $(s, s^{-1})$. They are equal to powers of $a^p b$ in $G_{p,q}$.

*Type 2.* These are the subwords of the form $t \dots t^{-1}$ which are determined by the inverse pairs $(t, t^{-1})$. They are equal to powers of $a^p b^{-1}$ in $G_{p,q}$.

*Type 3.* These are the subwords of the form $s^{-1} \dots s$ which are determined by the inverse pairs $(s^{-1}, s)$. They are equal to powers of $a$ in $G_{p,q}$.

*Type 4.* These are the subwords of the form $t^{-1} \dots t$ which are determined by the inverse pairs $(t^{-1}, t)$. They are equal to powers of $a$ in $G_{p,q}$.

*Type 5.* These are the remaining subwords of $w$. They are of the form $a^m$ for some $m \in \mathbb{Z}$.

A key result in this proof is is the following proposition.

**Proposition 4.3.5.** *[6, p. 1060] Let $w$ be a word of length $l$ in the generators $a^{\pm 1}, s^{\pm 1}, t^{\pm 1}$, and let $\alpha = \log_2(\frac{2p}{q})$. We then have the following:*

1. *If $w$ represents $a^m$ in $G_{p,q}$, then $\ell(m) \leq l^\alpha$.*
2. *If $w$ represents either $(a^p b)^m$ or $(a^p b^{-1})^m$, then $q\ell(m) \leq l^\alpha$.*

**Corollary 4.3.6.** *The distortion function of the torus subgroup $\langle a, b \mid [a, b] = 1\rangle \subset G_{p,q}$ is equivalent to $n^\alpha$, where $\alpha = \log_2(\frac{2p}{q})$.*

***Proof.*** By the proof of Theorem 4.2.2, we have that $w^k = a^{m_k}$, where $m_k \geq q\left(\frac{2p}{q}\right)$. Using that $\ell(w_k) \leq (4q)2^k$ and rearranging gives us that $m_k \geq \left(\frac{q}{(4q)^\alpha}\right)\ell(w_k)^\alpha$, and so the distortion function $d \geq n^\alpha$.

For the lower bound on distortion, notice that for each $g$ in the torus subgroup $\langle a, b \mid [a, b] = 1\rangle$, there are unique integers $r$ and $s$ such that $g = a^r (a^p b)^s$ in $\langle a, b \mid [a, b] = 1\rangle$. If $A$ is the generating set $\{a, a^p b\}$, then in the associated word metric $d_A$ we have that $d_A(1, g) = |r| + |s|$. Comparing this to the word metric $d_G$ corresponding to the generators $\{a, s, t\}$ for $G_{p,q}$, we have that

$$\begin{aligned} d_G(1, g)^\alpha &\geq d_G(1, a^r)^\alpha + d_G(1, (a^p b)^s)^\alpha && \text{(by triangle and Hölder inequalities)} \\ &\geq |r| + q|s| && \text{(by Proposition 4.3.5)} \\ &\geq d_A(1, g) \end{aligned}$$

By this, the distortion $d$ of the torus subgroup in $G_{p,q}$ satisfies $d(n) \leq n^\alpha$. Together, the lower and upper bound give equivalence of the distortion to $n^\alpha$, for $\alpha = \log_2(\frac{2p}{q})$. □

To prove our main result of the upper bound, we assume the following lemma without proof. For details, see [6, p. 1065].

**Lemma 4.3.7** (Shuffling Lemma). *Let $A$ be the free abelian group on the generating set $\{a, b_1, \ldots, b_k\}$. If $W_1, \ldots, W_r \in A$ are words of minimal length such that $W_1 \cdots W_n \underset{A}{=} a^n$ for some $n \in \mathbb{Z}$, then the product $W_1 \cdots W_n$ can be transformed to $a^n$ in at most $2k \sum_{i<j} \ell(W_i)\ell(W_j)$ steps as follows. Here, a step is an application of the commutator relations $[b_i, b_j]$ and $[a, b_i]$, for $1 \le i, j \le k$.*

**Proposition 4.3.8.** *Let $p \ge q$ be integers and consider the presentation of $G_{p,q}$ which we recall from Definition 4.1.4:*

$$G_{p,q} = \left\langle a, b, s, t \,\middle|\, [a, b] = 1, sa^q s^{-1} = a^p b, ta^q t^{-1} = a^p b^{-1} \right\rangle.$$

*There exists a constant $C \ge 1$ such that for all $m \in \mathbb{Z}$, all words $w$ in the generators $a^{\pm 1}, s^{\pm 1}, t^{\pm 1}$ and all words $v$ of the form $a^m$ or $(a^p b^{\pm 1})^m$, the following condition holds: If $w = v$ in $G_{p,q}$, then $A(wv^{-1}) \le C\ell(w)^{2\alpha}$, where $\alpha = \log_2 \left(\frac{2p}{q}\right)$.*

***Proof.*** We prove this by induction on the length of the word $w$. Call this length $l = \ell(w)$. Here we will show that we can take $C = \frac{2(p+1)^4}{q}$.

We consider the initial stage of the induction where $l \le q + 1$. Suppose first that $v$ is a word of the form $a^m$, and $w = v$ in $G_{p,q}$. Then $wa^{-m}$ is a word equal to the identity in $G_{p,q}$, and hence by Britton's lemma, the word given by $wa^{-m}$ is not reduced. Thus there is a consecutive subsequence given by a pair $s, s^{-1}$ or $t, t^{-1}$ with a word in $a^q$ or $a^p b^{\pm 1}$ between them. By inspection, this word must occur in $w$. In the first instance, we may write $w = w_1 s v s^{-1} w_2$ in $G_{p,q}$, with $v = a^{nq}$ for some $n \in \mathbb{Z}$. It follows that we must have $n = 0$, and therefore a free cancellation so that $w \underset{G_{p,q}}{=} w_1 s s^{-1} w_2 \underset{G_{p,q}}{=} w_1 w_2$. Otherwise, $l \ge l(v) + 2 > q + 1$, which contradicts our starting assumption. The argument is similar in the other cases. Since $w$ freely reduces to $v$, we have that the area $A(wv^{-1}) = 0 \le C\ell(w)^{2\alpha}$.

For the inductive step, we consider two cases:

*Case 1.* The word $w$ is one of the five types listed in Definition 4.3.4.

*Case 2.* The word $w$ is a product of subwords $w_1 w_2 \ldots w_n$ of types 1 to 5, where $n \ge 2$.

For Case 1, we show the inductive step holds when $w$ is a word of type 1. The proof for the other types is similar.

Suppose $w$ is a word of length $l > q + 1$, and assume that our inductive hypothesis holds for up to and including $k = l - 1$, for some $k \in \mathbb{N}_{>q+2}$. If $w$ is type 1, then $w$ is of the form $sw's^{-1}$ for $w' \in F(\{a, s, t\})$. Next, call $v \equiv (a^p b)^m$ for some $m$. We then have that $w = v$ in the group $G_{p,q}$. It follows that if we write $v' \equiv a^{mq}$, we have $w' = v'$

in $G_{p,q}$. Then, as $\ell(w') = l - 2$, we have reduced the length of our word by 2, so by our inductive assumption we have that $A(w'v'^{-1}) \le C\ell(w')^{2\alpha}$. In addition, note that $sv's^{-1}v^{-1}$ is a product of $m$ conjugates of the relation $sa^q s^{-1}(a^p b)^{-1}$. Therefore,

$$\begin{aligned} A(wv^{-1}) &= A(sw's^{-1}v^{-1}) \\ &\le A(sw'v'^{-1}s^{-1}) + A(sv's^{-1}v^{-1}) \qquad \text{(by Lemma 2.1.10)} \\ &\le C\ell(w')^{2\alpha} + m. \end{aligned}$$

By Proposition 4.3.5, $m \le \ell(w)^\alpha$, and as $\ell(w') = \ell(w) - 2$ this gives

$$A(wv^{-1}) \le C(\ell(w) - 2)^{2\alpha} + \ell(w)^\alpha,$$

which by algebraic manipulation is less than $\ell(w)^{2\alpha}$. This proves the induction for Case 1.

Next, consider Case 2 where $w$ is a product of subwords $w_1 w_2 \dots w_n$ of types 1 to 5, for $n \ge 2$. By our characterisation of inverse pairs in these five types, we may assume that each subword $w_i$ is equal to a unique word of the form $a^{m_i}$ or $(a^p b^{\pm 1})^{m_i}$. For each $w_i$ denote this corresponding word by $v_i$. Now let $l = \ell(w_i)$. As before, $v \equiv v_1 \dots v_n$ is equal to $w$ in $G_{p,q}$. By the previous case, we have that for each $1 \le i \le n$, $A(w_i(v_i)^{-1}) \le Cl_i^{2\alpha}$. Observe that the word $wv^{-1}$ can be expressed as a product of null-homotopic words $\prod_{i=1}^n w_i v_i^{-1}$ and $\left(\prod_{j=1}^n v_j\right) v^{-1}$. By applying Lemma 2.1.10, we have

$$\begin{aligned} A(wv^{-1}) &= A\left(\left(\prod_{i=1}^n w_i v_i^{-1}\right)\left(\prod_{j=1}^n v_j\right) v^{-1}\right) \\ &\le A\left(\prod_{i=1}^n w_i v_i^{-1}\right) + A\left(\left(\prod_{j=1}^n v_j\right) v^{-1}\right) \\ &\le C\sum_{i=1}^n l_i^{2\alpha} + A\left(\left(\prod_{j=1}^n v_j\right) v^{-1}\right). \end{aligned}$$

To conclude the proof, we make the following claim and show it leads directly to our desired result.

*Claim* 4.3.9. If $C = \frac{2(p+1)^4}{q^2}$, then

$$A\left(\left(\prod_{j=1}^n v_j\right) v^{-1}\right) \le C\sum_{i=1}^n l_i^\alpha \left(\sum_{j \ne i} l_j^\alpha\right).$$

A proof of this inequality involves Proposition 4.3.5 and the Shuffling Lemma (see Lemma 4.3.7). For details, we refer the interested reader to [6, p. 1067]. Assuming the claim holds, we get

$$A(wv^{-1}) \leq C_i l_i^{2\alpha} + C \sum_{i=1}^{n} l_i^{\alpha} \sum_{j \neq i} l_j^{\alpha} = C \left( \sum_{i=1}^{n} l_i^{\alpha} \right)^2 \leq C l^2 \alpha,$$

where the final inequality arises from the Hölder inequality (Proposition 4.3.3).

□

It follows that taking $v$ as the empty word in Proposition 4.3.8 above yields the upper bound.

**Theorem 4.3.10** (Upper bound for Dehn function)**.** *Suppose $w$ written in the generators $a^{\pm 1}, s^{\pm 1}, t^{\pm 1}$ is null-homotopic in $G_{p,q}$. Then, $A(w) \leq C\ell(w)^{2\alpha}$, where $\ell(w)$ is the length of $w$ with respect to the generators and $\alpha = \log_2\left(\frac{2p}{q}\right)$. It then follows that the Dehn function satisfies the upper bound $\delta_{G_{p,q}} \leq n^{2\alpha}$.*

Combining Theorems 4.2.2 and 4.3.10 combine to give Theorem 4.0.1. Therefore, the snowflake groups give rise to a dense set of exponents for Dehn functions in $\mathbb{P}_{\geq 2}$, and finally this shows uniqueness of the Gromov gap.

## 5 Further Research

We briefly review some of the research that led to Brady-Bridson's paper [6], and some recent discoveries about the isoperimetric spectrum which have followed since. Some insights of previous works include Baumslag-Miller-Short (1993) and Bridson-Pittet (1994) who showed that all integral values were elements of $\mathbb{P}$. From there on, the surrounding literature aimed to find collections of elements in the $[2, \infty)$ portion of $\mathbb{P}$ with smaller and smaller gaps between the elements, until Brady and Bridson showed that the snowflake groups gave rise to a dense collection of numbers in $[2, \infty)$.

Since the early 2000s, there have been many more interesting results proved about the isoperimetric spectrum. We give a couple of examples.

One important question is: *for any $\alpha \in [1, \infty)$, can we determine whether or not $\alpha \in \mathbb{P}$, i.e. is $n^\alpha$ is a Dehn function?* While this question is currently open, we state a result which has made progress towards proving it.

**Theorem 5.0.1** (Sapir, Birget, Rips (2002))**.** *If a real number $\alpha > 4$ is computable*[4] *in time less than $2^{2^{Cm}}$ for some constant $C > 0$, then $n^\alpha$ is equivalent to the Dehn function of a finitely presented group.*

Examples of computable values of $\alpha > 4$ include $\pi + 1$, $e^2$ and all rational numbers $\frac{p}{q} > 4$.

Another angle on recent research into the isoperimetric spectrum concerns a $k$-dimensional isoperimetric spectrum, denoted $\mathbb{P}^{(k)}$, which is defined by using a corresponding $k$-dimensional Dehn function. This extends our theory on areas to $k$-dimensional volumes. Interestingly, there is no analogy to the Gromov gap in higher dimensions:

**Theorem 5.0.2** (Brady, Forester (2008))**.** *The $k$-*dimensional isoperimetric spectrum $\mathbb{P}^{(k)}$ *is dense in $[1, \infty)$ for $k > 2$.*

For further details on recent research, we refer the interested reader to [13, p. 3–4].

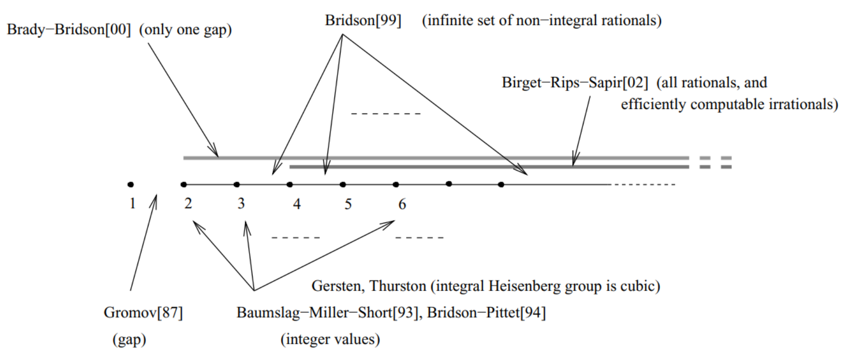


Figure 9: Diagram of recent results on the isoperimetric spectrum [3, p. 6].

[4]Being computable in time $T(n)$ means that there exists a Turing machine which, given $n$, computes a binary rational approximation of $\alpha$ with error at most $\frac{1}{2^{n+l}}$ in time at most $T(n)$.

# A Appendix

In this section, we present some useful results that complement our main discussion. Since these topics deviate slightly from our main topic of interest, we omit the proofs and instead refer to the relevant sections in Bridson and Haefliger [4] for details.

## A.1 Local geodesics

First, we introduce what it means for a path to be a $k$-local geodesic and state some useful properties.

**Definition A.1.1.** [4, p. 405] Let $X$ be a metric space and fix $k > 0$. A path $\gamma : [a, b] \to X$ is a *$k$-local geodesic* if $d(\gamma(t_1), \gamma(t_2)) = |t_2 - t_1|$ for all pairs $t_1, t_2 \in [a, b]$ such that $|t_2 - t_1| \leq k$.

In other words, a path $\gamma$ is a $k$-local geodesic if every subpath of $\gamma$ of length at most $k$ is geodesic.

**Theorem A.1.2.** *[4, p. 405–6] Let $X$ be a $\delta$-hyperbolic geodesic space and let $\gamma : [a, b] \to X$ be a $k$-local geodesic, where $k > 8\delta$. Then $\mathrm{Img}(\gamma)$ is contained in the $2\delta$-neighbourhood of any geodesic segment $[\gamma(a), \gamma(b)]$ connecting the endpoints of $\gamma$.*

We assume the definition of a *quasi-isometric embedding* from *MA4H4 Geometric Group Theory*, and recall that hyperbolicity is quasi-invariant. We refer the reader to the lecture notes of *MA4H4 Geometric Group Theory* for a proof.

**Theorem A.1.3.** *[4, p. 402] Let $X$ and $X'$ be geodesic metric spaces and let $f : X' \to X$ be a quasi-isometric embedding. If $X$ is hyperbolic, then $X'$ is hyperbolic.*

## A.2 Metric graphs

We next include a result that allows us to replace length spaces, possibly very complicated ones, by quasi-isometric metric graphs. This can be helpful when working with quasi-isometry invariants (such as Gromov hyperbolicity) because it allows one to use induction on the length of paths. This method used in the main proof of Section 3.2.

**Proposition A.2.1.** *[4, p. 152] There exist universal constants $\alpha$ and $\beta$ such that there is an $(\alpha, \beta)$-quasi-isometry from any length space to a metric graph, all of whose edges have length one.*

**Proposition A.2.2.** *[4, p. 415] Let $X'$ and $X$ be quasi-isometric length spaces. If there exists $f : [0, \infty) \to [0, \infty)$ and $\varepsilon > 0$ such that every loop in $X$ has an $\varepsilon$-filling, and $\mathrm{Area}_\varepsilon(\gamma) \leq f(l(\gamma))$ for every rectifiable loop $\gamma$ in $X$, then there exists $f' : [0, \infty) \to [0, \infty)$ such that $f'$ is a coarse isoperimetric bound for $X'$ such that $f' \leq f$.*

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